Properties of $\Sigma^{\Sigma^{(-)}}$-algebras in $\mathcal{E}qu$

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Category Theory 2017

Equilogical spaces (Scott, 1996)

$\mathcal{Equ}$: category of equilogical spaces and equivariant maps.

• **Objects**: An **equilogical space** is a triple $E = (|E|, \tau_E, \equiv_E)$, where $(|E|, \tau_E)$ is a T_0-space and $\equiv_E$ is an equivalence relation on $|E|$.

• **Morphisms**: An **equivariant map** $[f] : E \to F$ between the equilogical spaces E and F is an equivalence class of continuous functions $f\colon (|E|, \tau_E) \to (|F|, \tau_F)$ which preserve the relations, *i.e.*

$$\text{for all } x, x' \in |E|, \text{ if } x \equiv_E x', \text{ then } f(x) \equiv_F f(x').$$

Two such functions, $f, f'\colon E \to F$ are equivalent if for all $x \in |E|$, $f(x) \equiv_F f'(x)$.

D.S. Scott, *A new category? Domains, spaces and equivalence relations.* Manuscript, (1996).

Relevant subcategories of $\mathcal{E}qu$

$\mathcal{E}qu$ is a locally cartesian closed extension of $\mathcal{T}op_0$:

$$\begin{array}{ccc} \mathcal{T}op_0 & \xhookrightarrow[\text{full}]{Y} & \mathcal{E}qu \\ (X, \tau_X) & \longmapsto & (X, \tau_X, =) \end{array}$$

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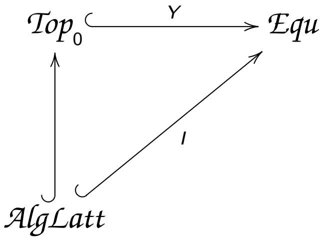

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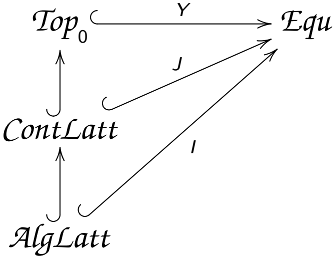

$\mathcal{A}lg\mathcal{L}att$: category of algebraic lattices and Scott-continuous functions.

$\mathcal{C}ont\mathcal{L}att$: category of continuous lattices and Scott-continuous functions.

$\mathcal{PE}qu$: category of partial equilogical spaces

• **Objects**: a partial equilogical space is a triple $A = (|A|, \tau_{Sc}, \approx_A)$, where $|A|$ is an algebraic lattice, τ_{Sc} is the Scott topology on $|A|$ and $\approx_A$ is a **partial equivalence relation** on $|A|$, namely it is symmetric and transitive, but not necessarily reflexive.

• **Morphisms**: $[f] : A \to B$ is an equivalence class of Scott-continuous functions $f : (|A|, \tau_{Sc}) \to (|B|, \tau_{Sc})$ such that

$$\text{for all } a, a' \in |A|, \text{ if } a \approx_A a', \text{ then } f(a) \approx_B f(a').$$

Two such functions $f, f' : A \to B$ are equivalent if for all $a, a' \in |A|$, if $a \approx_A a'$, then $f(a) \approx_B f'(a')$.

Theorem (Scott, 1996)

$\mathcal{E}qu$ and $\mathcal{PE}qu$ are equivalent

Products and exponentials in $\mathcal{PE}qu$

Let A and B be objects in $\mathcal{PE}qu$. Then

- their product $A \times B$ is

$$(|A| \times |B|, \tau_{Sc}, \approx_{A\times B})$$

  where $(a, b) \approx_{A\times B} (a', b')$ if $a \approx_A a'$ and $b \approx_B b'$;

- their exponential B^A is

$$(|B|^{|A|}, \tau_{Sc}, \approx_{B^A})$$

  where

  - $|B|^{|A|}$ is the algebraic lattice (ordered by the pointwise order) of the Scott-continuous functions from $|A|$ to $|B|$
  - $f \approx_{B^A} f'$ if for all $a, a' \in |A|$, if $a \approx_A a'$, then $f(a) \approx_B f'(a')$.

Double-power monad on $\mathcal{E}qu$

Σ, the **Sierpinski space** : T_0-space consisting of two points, $\perp$ and $\top$, and the only non-trivial open set is $\{\top\}$.

Double-power monad on $\mathcal{Equ}$

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Consider the self-adjoint functor

$$\mathcal{E}qu \underset{\Sigma^{(-)}}{\overset{\Sigma^{(-)}}{\rightleftarrows}} \mathcal{E}qu^{\mathrm{op}} \quad (\bot)$$

and the double power of Σ, denoted by $\Sigma^{\Sigma^{(-)}}$ or Σ^2, gives rise to a monad on $\mathcal{E}qu$.

$\mathcal{C}^{\Sigma^2}$: category of the Eilenberg-Moore algebras for the monad $\Sigma^{\Sigma^{(-)}}$ on $\mathcal{C}$.

Interesting subcategories of $\mathcal{PE}qu$

$\mathcal{RE}qu$: full subcategory of $\mathcal{PE}qu$ consisting of those triples $A = (|A|, \tau_{Sc}, \equiv_A)$, where $\equiv_A$ is an equivalence relation on $|A|$.

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$\mathcal{SE}qu$: full subcategory of $\mathcal{PE}qu$ consisting of those triples $A = (|A|, \tau_{Sc}, \sim_A)$, where $\sim_A$ is a **subreflexive** relation on $|A|$, namely it is contained in the diagonal relation.

$$\begin{array}{ccc} \mathcal{SE}qu & \hookrightarrow & \mathcal{PE}qu \\ \cong & & \cong \\ \mathcal{T}op_0 & \underset{Y}{\hookrightarrow} & \mathcal{E}qu \end{array}$$

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$$\begin{array}{ccc} \mathcal{SEqu} & \hookrightarrow & \mathcal{PEqu} \\ ||| & & ||| \\ \mathcal{Top}_0 & \underset{Y}{\hookrightarrow} & \mathcal{Equ} \end{array}$$

Proposition

$S \in \mathcal{SEqu}$, $R \in \mathcal{REqu} \Rightarrow S^R \in \mathcal{SEqu}$ and $R^S \in \mathcal{REqu}$

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$$\mathcal{RE}qu \underset{\Sigma^{(-)}}{\overset{\Sigma^{(-)}}{\rightleftarrows}}_{\perp} \mathcal{SE}qu^{\mathrm{op}} \qquad \mathcal{SE}qu \underset{\Sigma^{(-)}}{\overset{\Sigma^{(-)}}{\rightleftarrows}}_{\perp} \mathcal{RE}qu^{\mathrm{op}}$$

Double-power monad on $\mathcal{E}qu$

- A topological space X is exponentiable if and only if Σ^X is a topological space.
- If X is a topological space, the global section functor, applied to the equilogical space Σ^X, gives the topology of X.
- The monad $\Sigma^{\Sigma^{(-)}}$ on $\mathcal{E}qu$ represents the algebraic theory of Σ in $\mathcal{E}qu$.
- Σ^2-algebras in $\mathcal{E}qu \rightsquigarrow$ frames.

P. Taylor, *Sober spaces and continuations.* Theory Appl. Categ., (2002).

P. Taylor, *Subspaces in abstract Stone duality.* Theory Appl. Categ., (2002).

S.J. Vickers and C.F. Townsend, *A universal characterization for the double power-locale.* Theoret. Comput. Sci., (2004).

Σ^2-algebras and frames

Every Σ^2-algebra in $\mathcal{E}qu$ inherits a frame structure from Σ, since the lattice operations $\wedge\colon \Sigma^2 \to \Sigma$ and $\vee\colon \Sigma^2 \to \Sigma$ are Scott-continuous and, in addition, for each set I, seen as a discrete topological space, $\vee^I\colon \Sigma^I \to \Sigma$ is Scott-continuous.

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So, the global section functor $\Gamma\colon \mathcal{E}qu \to \mathcal{S}et$ extends to a faithful functor

$$\begin{array}{ccc} \mathcal{E}qu^{\Sigma^2} & \xrightarrow{\ \Gamma\ } & \mathcal{F}rm \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{E}qu & \xrightarrow{\ \Gamma\ } & \mathcal{S}et \end{array}$$

Σ^2-algebras and frames

Every Σ^2-algebra in $\mathcal{Equ}$ inherits a frame structure from Σ, since the lattice operations $\wedge\colon \Sigma^2 \to \Sigma$ and $\vee\colon \Sigma^2 \to \Sigma$ are Scott-continuous and, in addition, for each set I, seen as a discrete topological space, $\vee^I\colon \Sigma^I \to \Sigma$ is Scott-continuous.

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So, the global section functor $\Gamma\colon \mathcal{Equ} \to \mathcal{Set}$ extends to a faithful functor

$$\begin{array}{ccc} \mathcal{Equ}^{\Sigma^2} & \xrightarrow{\Gamma} & \mathcal{Frm} \\ (E, \alpha) & \longmapsto & |E|/\equiv_E \end{array}$$

Uniqueness of the structure map

Theorem

The structure map on an object A in $\mathcal{RE}qu$, if exists, is unique

In particular, the claim holds for Σ^2-algebras in $\mathcal{A}lg\mathcal{L}att$.

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In particular, the claim holds for Σ^2-algebras in $\mathcal{AlgLatt}$.

Sketch of the proof:

- Each element of the algebraic lattice $|\Sigma^{(\Sigma^A)}|$ can be written as an arbitrary join of finite meets of functions of the form $\eta_A(k)$, where k is a compact element of $|A|$ and η is the unit of the monad.
- In every Σ^2-algebra of $\mathcal{REqu}$ which is a power of Σ, the frame structure coincides with the algebraic lattice structure.

$\mathcal{C}ont\mathcal{L}att \equiv \mathcal{RE}qu \cap \mathcal{SE}qu$

Recall that the objects in $\mathcal{T}op_0$ which are injective with respect to the topological inclusions are exactly continuous lattices endowed with the Scott topology.

Proposition

Let (X, τ_X) be a T_0-space. (X, τ_X) is injective if and only if $(X, \tau_X, =)$ is isomorphic to an object in $\mathcal{RE}qu$

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By uniqueness of the structure map in $\mathcal{RE}qu$, we can prove:

Corollary

The structure map on an object C in $\mathcal{C}ont\mathcal{L}att$, if exists, is unique.

A characterization for $\mathcal{ContLatt}^{\Sigma^2}$ and $\mathcal{AlgLatt}^{\Sigma^2}$

We denote with $\mathcal{ContFrm}$ and $\mathcal{AlgFrm}$ the full subcategories of $\mathcal{Frm}$ consisting of continuous frames and algebraic frames, respectively.

Theorem

The categories $\mathcal{ContLatt}^{\Sigma^2}$ and $\mathcal{ContFrm}$ are equivalent.

Corollary

The categories $\mathcal{AlgLatt}^{\Sigma^2}$ and $\mathcal{AlgFrm}$ are equivalent.

The equivalence is given by the following functors:

$$\mathcal{ContFrm} \underset{\Gamma}{\overset{\Sigma^{(-)}\circ \mathrm{pt}}{\rightleftarrows}} \mathcal{ContLatt}^{\Sigma^2}$$

(the same functors give the equivalence also in the algebraic case)

Σ^2-algebras in $\mathcal{Top}_0 \equiv \mathcal{SEqu}$

Theorem

If (X, α) is a Σ^2-algebra in $\mathcal{Top}_0$, then X is a compact, connected, sober topological space.

Theorem

There are Σ^2-algebras (X, α) in $\mathcal{Top}_0$ such that X is not locally compact.

Sketch of the proof:

Consider X a non-exponentiable topological space. Then, $\Sigma^{(\Sigma^X)}$ is a Σ^2-algebra in $\mathcal{Top}_0$ but it cannot be exponentiable. Indeed, if it is, $\Sigma^{(\Sigma^{(\Sigma^X)})}$ is an injective T_0-space and, since Σ^X is a retract in $\mathcal{Equ}$ of $\Sigma^{(\Sigma^{(\Sigma^X)})}$, Σ^X is a topological space and so X is exponentiable.

Σ^2-algebras in $\mathcal{REqu}$

Theorem

Each Σ^2-algebra in $\mathcal{REqu}$ is of the form (A, α) where $|A|$ is an algebraic frame.

Indeed, if (A, α) is a Σ^2-algebra in $\mathcal{REqu}$, then

$$A \cong (|\Sigma^{(\Sigma^A)}|, \tau_{Sc}, \equiv_{q_A \circ \alpha})$$

where $F \equiv_{q_A \circ \alpha} G$ if $\alpha(F) \equiv_A \alpha(G)$.

It is a consequence of a more general fact:

- if (E, α) is in $\mathcal{Equ}^{\Sigma^2}$, then E is isomorphic to an equilogical space $F = (|F|, \tau_F, \equiv_F)$ where $|F| = |\Sigma^{(\Sigma^E)}|$ is a frame.

Σ^2-algebras in $\mathcal{REqu}$

Let (A, α) be a Σ^2-algebra in $\mathcal{REqu}$.

$\Sigma^2(A, \Sigma)$: set of Σ^2-homomorphisms from A to Σ.

Let τ_e be the subspace topology on $\Sigma^2(A, \Sigma)$ induced by the inclusion

$$\Sigma^2(A, \Sigma) \xhookrightarrow{e} (|\Sigma^A|, \tau_{Sc})$$

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$\mathcal{Frm}(|A|/\equiv_A, \Sigma)$: set of frame homomorphisms from $|A|/\equiv_A$ to Σ.
Let τ_i be the subspace topology on $\mathcal{Frm}(|A|/\equiv_A, \Sigma)$ with respect to the inclusion

$$\begin{array}{ccc} \mathcal{Frm}(|A|/\equiv_A, \Sigma) & \xhookrightarrow{i} & (|\Sigma^A|, \tau_{Sc}) \\ p & \longmapsto & p' : a \mapsto p([a]) \end{array}$$

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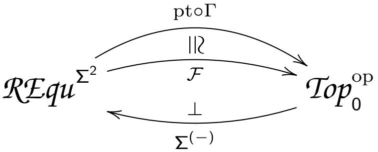
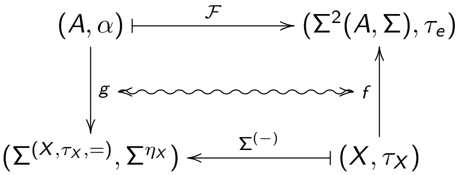
Σ^2-algebras in $\mathcal{REqu}$

Theorem

There is an adjunction between the categories $\mathcal{REqu}^{\Sigma^2}$ and $\mathcal{Top}_0^{\mathrm{op}}$.





Σ^2-algebras in $\mathcal{REqu}$

Theorem

Objects in $\mathcal{REqu}^{\Sigma^2}$ of the form Σ^S, where S is in $\mathcal{SEqu}$, are exactly those Σ^2-algebras (A, α) such that $|A|/\equiv_A$ is a **spatial** frame.

Indeed:

- If (X, τ_X) is a T_0-space, then $|\Sigma^X|/\equiv_{\Sigma^X} \cong \tau_X$.
- If $|A|/\equiv_A$ is spatial, then

$$(A, \alpha) \cong \left(\Sigma^{(\Sigma^2(A,\Sigma),\tau_e,=)}, \Sigma^{\eta_{\Sigma^2(A,\Sigma)}}\right)$$

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Corollary

The adjunction restricts to an equivalence between the subcategory $\mathcal{Sob}^{\mathrm{op}}$ of $\mathcal{Top}_0^{\mathrm{op}}$ consisting of sober spaces and the subcategory of $\mathcal{REqu}^{\Sigma^2}$ consisting of those Σ^2-algebras (A, α) such that $|A|/\equiv_A$ is spatial.