Coherently closed tangent categories

The link between SDG and the ∂-λ-calculus

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CCCs for differential geometry

There are two main approaches that have appeared for axiomatizing differential geometry that is meant to be performed in a CCC.

Synthetic differential geometry

- Led to drastic simplifications of many parts of differential geometry
- Also greatly simplified constructions by using internal logic
- Led to "previously undreamed of" opportunities

The differential λ-calculus

- The simply typed version led to monoidal differential categories
- Then Cartesian differential categories
- And ultimately to tangent categories

Synthetic differential geometry

Lawvere ‘67:

- Need a ring R and an ideal $D \subseteq R$ such that $d^2 = 0$ for all $d \in D$;
- The map $R \times R \longrightarrow [D, R]$ given by $(a, b) \mapsto \lambda d.d \cdot a + b$ must be an isomorphism;
- Need spaces M that believe that D is well behaved. For example, all spaces must believe that

$$\begin{array}{ccc} 1 & \xrightarrow{\;0\;} & D \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle i_1} \\ D & \xrightarrow[\;i_0\;]{} & D(2) \end{array}$$

where $D(2) = \{(x, y) | x^2 = y^2 = xy = 0\}$ is a pushout.
- D is so small that $[D, \_]$ has a right adjoint.

Differential semantics of λ-calculi

Wanted: a model of linear logic where

- Proofs are interepreted as smooth (or at least continuous) functions
- Linear proofs are interpreted as linear proofs in a mathematical sense

So that linearity (proof theory) is captured mathematically (by being represented by a line).

One can then obtain a linear (proof theory) approximation to a proof...

In the models, this turned out to be a differential operator [3, 2]

The differential λ-calculus

Interestingly, the differential operator is a special cut or composition:

v : Abstract vector quantity

$$\frac{\Gamma, x : A \vdash m : B \quad \Gamma \vdash a : A \quad \Gamma \vdash v : A}{\Gamma \vdash \frac{\mathrm{d}m}{\mathrm{d}x}(a) \cdot v : B}$$

Determines how v can be used linearly

Importantly, this differential composition operator is just composition with a tangent vector v. This is reminiscent of SDG:

$$T(f)(v) := D \xrightarrow{v} M \xrightarrow{f} N$$

The differential λcalculus

Important theorems:

Theorem (Ehrhard-Regnier)

The equational theory of the λ_∂ may be oriented into a rewriting system that is confluent modulo.

Corollary (Ehrhard-Regnier)

λ_∂ is a conservative extension of λ.

Slogan: The λ-calculus was smooth all along!

Today's talk

- These two formal systems are more intimately related than the metaphor of linear composition
- A link will be established using abstract tangent structure studied by [4] and [1]

Abstract tangent structure

A tangent structure on $\mathbb{X}$ is an endofunctor, that sends an object M, naturally, to a commutative monoid in $\mathbb{X}/M$, together with two additional natural transformations.

Tangent structure

Axiom 1 [Additive bundle]: A category has an **additive bundle structure**, when there is an endofunctor $\mathbb{X} \xrightarrow{T} \mathbb{X}$.

There must be a natural transformation $$\begin{array}{c} TM \\ \big\downarrow{\scriptstyle p} \\ M \end{array}$$

There must be natural $+, 0$ giving p the structure of a commutative monoid in $\mathbb{X}/M$.

Tangent structure

A **transverse system** is a collection of pullback squares in a category $\mathcal{Q}$. An endofunctor $\mathbb{X} \xrightarrow{T} \mathbb{X}$ is **transverse** when $T(q) \in \mathcal{Q}$ for all $q \in \mathcal{Q}$.

Axiom 2 [Transerversality]: The tangent functor must be transverse for some transverse system containing the pullback powers of p and another pullback shown below.

The idea is that only pullbacks of transverse maps behave well.

Tangent structure

As T preserves transverse pullbacks and equations, T^2M is a commutative monoid in $\mathbb{X}/TM$ in two different ways:

$$\begin{array}{ccc} T^2M & \qquad & T^2M \\ {\scriptstyle p_{TM}}\downarrow & & \downarrow{\scriptstyle T(p_M)} \\ TM & & TM \end{array}$$

Tangent structure

Axiom 3 [Canonical flip]: There is an involution

$$\begin{array}{ccc} T^2M & \xrightarrow{\quad c \quad} & T^2M \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle Tp} & \\ & TM & \end{array}$$

that exchanges the addition.

Tangent structure

Axoim 4 [Vertical lift]: There is a lift:

$$\begin{array}{ccc} TM & \xrightarrow{l} & T^2M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Tp} \\ M & \xrightarrow[0]{} & TM \end{array}$$

that lifts the additive structure of p to the additive structure of Tp.

Tangent structure

Axiom 5 [Coherence]:
l and c form an internally commutative cosemigroup in $[\mathbb{X}, \mathbb{X}]$ Add string diagrams

Tangent structure

Axiom 6 [Universality of lift]:

$$\begin{array}{ccccc} T_2M & \xrightarrow{l\times 0} & T_2^2M & \xrightarrow{T(+)} & T^2M \\ {\scriptstyle \pi_0 p}\downarrow & & & & \downarrow{\scriptstyle Tp} \\ M & & \xrightarrow[0]{\hspace{6em}} & & TM \end{array}$$

is a transverse pullback.

Tangent structure

A **tangent category** is a category with an endofunctor $\mathbb{X} \xrightarrow{T} \mathbb{X}$ that satisfies axioms 1-6.

Examples of tangent categories

- Smooth manifolds. The usual tangent bundle of a manifold.
- Finitely presented affine schemes. Not generally cartesian closed; however, $\mathbb{Z}[x]/x^2$ is exponentiable. $TA := [\mathbb{Z}[x]/x^2, A]$.
- **Any model of synthetic differential geometry.**
- Any cartesian differential category.
- **The classifying category of the differential λ-calculus.**
- The idempotent splitting of any tangent category.
- The manifold construction applied to a tangent category.
- Presheaves
- Functor categories into a tangent category.

SDG and λ_∂ part 1

The differential λ-calculus provides the internal logic of Euclidean vector spaces in SDG.

Differential objects

- SMan contains the category CartSp
- A category of microlinear spaces contains the category of euclidean vector spaces
- Both cases: determined as monoids V for which $TV \xrightarrow{p} V$ has a left projection in a product diagram

$$V \xleftarrow{\hat{p}} TV \xrightarrow{p} V$$

- Objects that behave like vector spaces are called differential objects.

Differential Objects

An object V is a **differential object** when there is a map $TV \xrightarrow{\hat{p}} V$, so product, and it is a commutative monoid (V, σ, ζ), such that

$$
\begin{array}{ccc}
V & \xrightarrow{0} & TV \\
{\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \hat{p}} \\
1 & \xrightarrow[\zeta]{} & V
\end{array}
\qquad
\begin{array}{ccc}
T_2 V & \xrightarrow{+} & TV \\
{\scriptstyle \hat{p}\times\hat{p}}\downarrow & & \downarrow{\scriptstyle \hat{p}} \\
V \times V & \xrightarrow[\sigma]{} & V
\end{array}
\qquad
\begin{array}{ccc}
TV & \xrightarrow{\hat{p}} & V \\
{\scriptstyle l}\downarrow & & \uparrow{\scriptstyle \hat{p}} \\
T^2 V & \xrightarrow{T\hat{p}} & TV
\end{array}
$$

$$
\begin{array}{ccccc}
T(V \times V) & \xrightarrow{\langle \pi_0, \pi_1 \rangle} & TV \times TV & \xrightarrow{\hat{p}\times\hat{p}} & V \times V \\
{\scriptstyle T\sigma}\downarrow & & & & \downarrow{\scriptstyle \sigma} \\
TV & & \xrightarrow[\hat{p}]{\qquad\qquad\qquad} & & V
\end{array}
$$

The differential of differential objects

There is a ‘combinator’

$$\frac{A \xrightarrow{f} B}{D[f] := A \times A \xrightarrow[\simeq]{} T(A) \xrightarrow[T(f)]{} T(B) \xrightarrow[\hat{p}]{} B}$$

linear

Theorem

The differential objects of a tangent category are always a Cartesian differential category.

Strong tangent categories

A tangent category with finite products is **cartesian** when pullbacks along $A \xrightarrow{!} 1$ are transverse.

In particular, we have $T(A) \times T(B) \simeq T(A \times B)$.

Then tangent categories are strong by:

$$\theta := A \times TB \xrightarrow{0 \times 1} TA \times TB \xrightarrow{\simeq} T(A \times B)$$

Closed tangent categories

When $\mathbb{X}$ is a closed tangent category, strength gives:

$$\frac{A \times T[A,B] \xrightarrow{\theta} T(A \times [A,B]) \xrightarrow{T\mathsf{ev}} TB}{T[A,B] \xrightarrow[\psi = \lambda(\theta T\mathsf{ev})]{} [A, TB]}$$

as an **exponentiable strength**.

Strength

Moreover, all the structural transformations are strong and exponentially strong.

$$\begin{CD} T[A,B] @>p>> [A,B] \\ @V{\psi}VV @| \\ [A,TB] @>>{[A,p]}> [A,B] \end{CD} \qquad \begin{CD} T[A,B] @>l>> T^2[A,B] \\ @. @VV{T(\psi)}V \\ @V{\psi}VV T[A,TB] \\ @. @VV{\psi}V \\ [A,TB] @>>{[A,l]}> [A,T^2B] \end{CD}$$

$$\begin{CD} T^2[A,B] @>c>> T^2[A,B] \\ @V{T(\psi)\psi}VV @VV{T(\psi)\psi}V \\ [A,T^2B] @>>{[A,c]}> [A,T^2B] \end{CD} \qquad \begin{CD} T_2[A,B] @>+>> T[A,B] \\ @V{\psi_2}VV @VV{\psi}V \\ [A,T_2B] @>>{[A,+]}> [A,T_2B] \end{CD} \qquad \begin{CD} [A,B] @>0>> T([A,B]) \\ @| @VV{\psi}V \\ [A,B] @>>{[A,0]}> [A,T(B)] \end{CD}$$

Coherently differential closed tangent categories

When V is a monoid, so is $[M, V]$.

Differential objects are **additively coherent** when the canonical monoid structure, is the only one.

Proposition

In a cartesian closed category with additively coherent monoids, the differential objects are a cartesian closed left additive category.

Corollary

In this case, the differential objects model the algebraic λ-calculus.

Coherently differential closed tangent categories

The differential objects are **differentially coherent** when the strength

$$T[A, B] \longrightarrow [A, TB]$$

is an isomorphism.
Differential coherence implies additive coherence

Proposition

The classifying category $\mathcal{C}[\lambda_\partial]$ may given the structure of a tangent category with coherent differential objects.

Proposition

The differential objects of a differentially coherent tangent category are a model of the differential λ-calculus.

Euclidean vector spaces

Corollary

The only models of the differential λ-calculus arise as the differential objects of a closed tangent category with coherent differential objects.

Euclidean vector spaces

Representable tangent category: $TM = [D, M]$

$$T[M, V] = [D, [M, V]] \xrightarrow{\psi} [M, [D, V]] = [M, TV]$$

is the swap isomorphism

Corollary

The Euclidean vector spaces in SDG are a model of the differential λ-calculus.

Corollary

Convenient vector spaces are a model of the differential λ-calculus.

The simple slice and global coherent tangent categories

Proposition

The strength $A \times TB \xrightarrow{\theta} T(A \times B)$ *is a comonad lifting law:*

$$T; (A \times \_) \xrightarrow{\theta} (A \times \_); T$$

Corollary

The tangent functor lifts to each coKleisli category, $\mathbb{X}[A]$.

The simple slice and global coherent tangent categories

Proposition

Each of the structural transformations lifts against θ.

The strength of the structure transformations guarantees they lift.

Corollary

All the structure transformations lift to each $\mathbb{X}[A]$

The simple slice and global coherent tangent categories

Corollary

When $\mathbb{X}$ is a tangent category, each $\mathbb{X}[A]$ is a tangent category.

Explicitly,

$$\dfrac{\dfrac{B \xrightarrow{f} C \qquad \mathbb{X}[A]}{A \times B \xrightarrow{f} C \qquad \mathbb{X}}}{\dfrac{A \times TB \xrightarrow{\theta} T(A \times B) \xrightarrow{Tf} TC \qquad \mathbb{X}}{TB \xrightarrow[\theta T(f)]{} TC \qquad \mathbb{X}[A]}}$$

The simple slice and global coherent tangent categories

When $\mathbb{X}$ is closed, the simple slice may be presented as the Kleisli category for $[A, \_]$.

Proposition

When $\mathbb{X}$ is a closed tangent category, for which the exponential strength:

$$T[A, B] \xrightarrow{\psi} [A, TB]$$

is an isomorphism, then $\mathbb{X}^A$ is an equivalent tangent category to $\mathbb{X}[A]$.

When ψ is an isomorphism, a tangent category is **coherently closed**.

SDG and λ_∂ part 2

The differential λ-calculus provides the internal logic of Euclidean vector bundles in SDG.

Locally Coherently Closed Tangent Categories

Euclidean vector spaces in SDG $\rightsquigarrow$ differential λ-calculus.

1. Euclidean vector bundles are a central focus in differential geometry
2. Locally, they are Euclidean vector spaces...
3. We would like to get at these!

Locally Coherently Closed Tangent Categories

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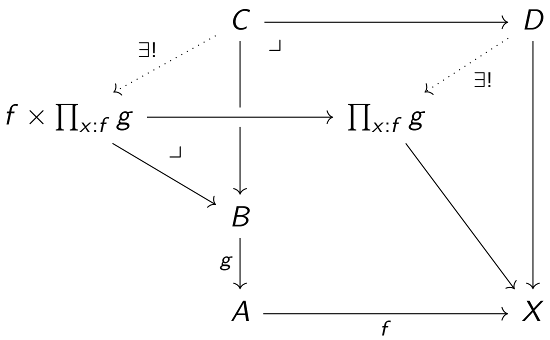
Locally means in a slice category

Locally Cartesian Closed Categories

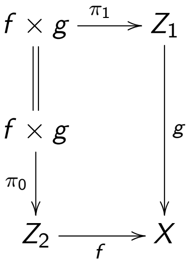
A map is $A \xrightarrow{f} X$ is exponentiable when for all $B \xrightarrow{g} A$ there is a terminal pullback extension [5]:



Locally Cartesian Closed Categories

A category is **locally cartesian closed** when it has finite limits, and every map is exponentiable.

This yields a Cartesian closed structure in each slice by taking the terminal diagram of



yields $[f, g]$.

Locally Coherently Closed Tangent Categories

Need:

- The slice of a tangent category by M is a tangent category with T_M
- The coherence isomorphism $T_M[A, B] \simeq [A, T_M B]$ in every slice

Locally Coherently Closed Tangent Categories

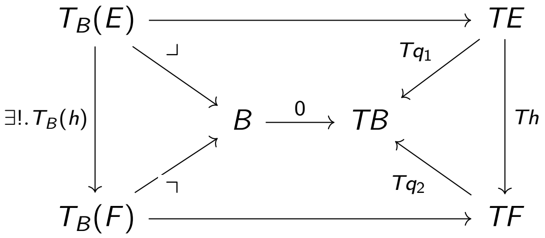
Theorem (CC'14)

When a tangent category has finite limits[1]*, the slice is always a tangent category.*

[1] can be weakened

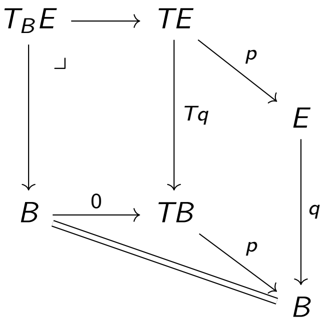
Slice tangent structure

The slice tangent structure in $\mathbb{X}/B$ is given by pullback along $B \xrightarrow{0} TB$. For $T_B(h)$:



Slice tangent structure

The slice tangent structure in $\mathbb{X}/B$ is given by pullback along $B \xrightarrow{0} TB$. For $p_B : T_B(E) \to E$:



Differential bundles

Theorem (CC'14)

Suppose $\mathbb{X}$ has finite limits.
Differential objects in the tangent category $\mathbb{X}/B$ are precisely differential bundles over B in $\mathbb{X}$.

This theorem says that in fact differential bundles really do provide local linear algebra.

It also formally reduces what is needed to: $T_M[A, B] \simeq [A, T_M B]$ in the slice over M.

Locally Coherently Closed Tangent Categories

Proposition

When a tangent functor satisfies

$$T\left[\prod_{x:A} B\right] \simeq \left[\prod_{x:A} TB\right]$$

then in each slice it has

$$T_M[A, B] \simeq [A, T_M B]$$

Follows from Beck-Chevalley

A locally cartesian closed tangent category is **locally coherently closed** when $T[\prod_{x:A} B] \simeq \prod_{x:A} TB$

Locally Coherently Closed Tangent Categories

$$T_M[q_E, q_F]$$

Locally Coherently Closed Tangent Categories

$$T_M[q_E, q_F]$$
$$\equiv 0^*(T(\prod q_E e_F))$$

Locally Coherently Closed Tangent Categories

$$
\begin{aligned}
&T_M[q_E, q_F] \\
&\equiv 0^*(T(\prod q_E e_F)) \\
&\simeq 0^*(\prod q_E(Tq_F))
\end{aligned}
$$

Locally Coherently Closed Tangent Categories

$$
\begin{aligned}
&T_M[q_E, q_F] \\
&\equiv 0^*(T(\prod q_E e_F)) \\
&\simeq 0^*(\prod q_E(Tq_F)) \\
&\simeq \prod q_E(0^* T(q_F))
\end{aligned}
$$

Locally Coherently Closed Tangent Categories

$$
\begin{aligned}
&T_M[q_E, q_F] \\
&\equiv 0^*(T(\prod q_E e_F)) \\
&\simeq 0^*(\prod q_E(Tq_F)) \\
&\simeq \prod q_E(0^* T(q_F)) \\
&\equiv [q_E, T_M q_F]
\end{aligned}
$$

Locally coherently closed tangent categories

Thus

Corollary

Suppose $\mathbb{X}$ has finite limits and is locally coherently closed, then each $\mathbb{X}/B$ is a coherently closed tangent category. The differential objects of $\mathbb{X}/B$ is a model of the differential λ-calculus.

Euclidean Vector Bundles

Proposition

In a locally cartesian closed tangent category, the category of differential bundles over B is a model of the differential λ-calculus.

Hence,

Corollary

In any model of Nishimura's axiomatic differntial geometry, the category of Euclidean vector bundles over B are a model of the differential λ-calculus.

Conclusion

- In SDG and SMan, Euclidean vector spaces are modelled by the differential λ-calculus.
- In SDG and SMan, Euclidean vector bundles are modelled by the differential λ-calculus.
- The more general reasoning in microlinear spaces is still needed for global reasoning about vector bundles.

Ongoing projects

- Formalizing more logic and differential geometry in bundle categories.
  - For example, local monoidal structure with duals...
- Formulating a direct syntax for all tangent categories
  - Including a syntax for the dependently typed differential λ-calculus.
  - For the latter, this is necessary – adding equalizers or identity types.
- Physics in tangent categories with substructual aspects of the logic in this talk...

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