Internal Neighbourhood Spaces

Partha Pratim Ghosh

Department of Mathematical Sciences, UNISA
Email: ghoshpp@unisa.ac.za

CT 2017, July 16-22 2017
University of British Columbia
Vancouver

In [Bentley et al., 1991] the authors state in their *Introduction*:

[Bentley et al., 1991]
Bentley, L., Herrlich, H., and Lowen, R. (1991).
Improving constructions in topology.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 3–20. Heldermann, Berlin.

In [Bentley et al., 1991] the authors state in their *Introduction*:

> *…to obtain useful results, topologists have often forced* PROPERTIES OF SPACES OR MAPS *in the sense of adding supplementary and extraneous conditions or even changing the definition of a property altogether, and left* CATEGORICALLY DEFINED CONSTRUCTIONS *well alone, i.e., essentially continued working in* $\boldsymbol{Top}$*.*
> *Our aim in this paper is to provide evidence that doing precisely the opposite, i.e., leaving concepts as they are but stepping outside* $\boldsymbol{Top}$ *and thereby changing constructions in an appropriate way will illuminate the situation and provide a natural setting or solution for problems for which no decent solution in* $\boldsymbol{Top}$ *or any reasonable subcategory of* $\boldsymbol{Top}$ *seems to exist. …*

[Bentley et al., 1991]

Bentley, L., Herrlich, H., and Lowen, R. (1991).
Improving constructions in topology.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 3–20. Heldermann, Berlin.

Extensions of $\boldsymbol{Top}$

$$
\begin{tikzcd}
\boldsymbol{psTop} \arrow[drr, "\text{topological}"] & \boldsymbol{prTop} \arrow[l, "\text{reflective}"'] \arrow[dr, "\text{topological}"] & \boldsymbol{Top} \arrow[l, "\text{reflective}"'] \arrow[d, "\text{topological}"] \\
 & & \boldsymbol{Set}
\end{tikzcd}
$$

Extensions of $\boldsymbol{Top}$

$$
\begin{tikzcd}
\boldsymbol{psTop} \arrow[drr, "\text{topological}"] & \boldsymbol{prTop} \arrow[l, "\text{reflective}"'] \arrow[dr, "\text{topological}"] & \boldsymbol{Top} \arrow[l, "\text{reflective}"'] \arrow[d, "\text{topological}"] \\
& & \boldsymbol{Set}
\end{tikzcd}
$$

$\boldsymbol{prTop}$ is the category of pretopological spaces

Extensions of $\boldsymbol{Top}$

$$
\begin{tikzcd}
\boldsymbol{psTop} \arrow[drr, "\text{topological}"] & & \boldsymbol{prTop} \arrow[ll, "\text{reflective}"'] \arrow[dr, "\text{topological}"] & \boldsymbol{Top} \arrow[l, "\text{reflective}"'] \arrow[d, "\text{topological}"] \\
 & & & \boldsymbol{Set}
\end{tikzcd}
$$

$\boldsymbol{prTop}$ is the category of pretopological spaces
$\boldsymbol{psTop}$ is the category of pseudotopological spaces

Purpose of this Talk

Extend this construction to a more general context of a *well behaved* finitely complete category $\mathbb{A}$.

Purpose of this Talk

$$
\begin{tikzcd}
\boldsymbol{pse}[\mathbb{A}] \arrow[ddrr, bend right, "\text{topological}"'] & \boldsymbol{pre}[\mathbb{A}] \arrow[l, "\text{reflective}"'] \arrow[ddr, bend right, "\text{topological}"'] & \boldsymbol{wNbd}[\mathbb{A}] \arrow[l, "\text{reflective}"'] \arrow[dd, "\text{topological}"] & \boldsymbol{Nbd}[\mathbb{A}] \arrow[l, "\text{non-full subcategory}"'] \arrow[dd, bend left, "\text{topological}"] \\
 & & & \boldsymbol{Top}[\mathbb{A}] \arrow[u, "\text{reflective}"] \arrow[d, "\text{topological}"] \\
 & & \mathbb{A} & \mathbb{A}_{\boldsymbol{ppj}} \arrow[l, "\text{non-full subcategory}"]
\end{tikzcd}
$$

Purpose of this Talk

$$
\begin{tikzcd}
\boldsymbol{pse}[\mathbb{A}] \arrow[ddrr, bend right, "\text{topological}"'] & \boldsymbol{pre}[\mathbb{A}] \arrow[l, "\text{reflective}"'] \arrow[ddr, bend right, "\text{topological}"'] & \boldsymbol{wNbd}[\mathbb{A}] \arrow[l, "\text{reflective}"'] \arrow[dd, "\text{topological}"] & \boldsymbol{Nbd}[\mathbb{A}] \arrow[l, "\text{reflective}"'] \arrow[d, equal, "\text{in a special context including when } \mathbb{A} = \boldsymbol{Set}"'] \arrow[dd, bend left=60, "\text{topological}"] \\
 & & & \boldsymbol{Top}[\mathbb{A}] \arrow[d, "\text{topological}"] \\
 & & \mathbb{A} \arrow[r, equal] & \mathbb{A}_{\boldsymbol{ppj}}
\end{tikzcd}
$$

Topological Spaces I

Theorem 2.1

Given any set X and a function $X \xrightarrow{\xi} \boldsymbol{Fil}(X)$ from the set X to the set $\boldsymbol{Fil}(X)$ of all filters on X with the properties:

$$U \in \xi(x) \Rightarrow x \in U \tag{1}$$

and

$$U \in \xi(x) \Rightarrow (\exists V \in \xi(x))\big(y \in V \Rightarrow U \in \xi(y)\big) \tag{2}$$

there exists a unique topology Ξ on X such that for each $x \in X$, $\xi(x)$ is the set of all neighbourhoods of the point x in the topological space (X, Ξ).

Topological Spaces I

Theorem 2.1
Given any set X and a function $X \xrightarrow{\xi} \boldsymbol{Fil}(X)$ from the set X to the set $\boldsymbol{Fil}(X)$ of all filters on X with the properties:

$$U \in \xi(x) \Rightarrow x \in U \tag{1}$$

and

$$U \in \xi(x) \Rightarrow (\exists V \in \xi(x))\big(y \in V \Rightarrow U \in \xi(y)\big) \tag{2}$$

there exists a unique topology Ξ on X such that for each $x \in X$, $\xi(x)$ is the set of all neighbourhoods of the point x in the topological space (X, Ξ).

Moral:
There is a one-to-one correspondence between topologies on X and functions $X \xrightarrow{\xi} \boldsymbol{Fil}(X)$ satisfying (1), (2).

Understanding Condition (2)

$$U \in \xi(x) \Rightarrow (\exists V \in \xi(x))\big(y \in V \Rightarrow U \in \xi(y)\big)$$

Understanding Condition (2)

$$U \in \xi(x) \Rightarrow (\exists V \in \xi(x))\big(y \in V \Rightarrow U \in \xi(y)\big)$$

$$\begin{array}{ccccccc}
x_{11} & x_{12} & x_{13} & \cdots & x_{1n} & \cdots & \to p_1 \\
x_{21} & x_{22} & x_{23} & \cdots & x_{2n} & \cdots & \to p_2 \\
x_{31} & x_{32} & x_{33} & \cdots & x_{3n} & \cdots & \to p_3 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} & \cdots & \to p_m \\
\vdots & \vdots & \vdots & & \vdots & & \vdots
\end{array}$$

Understanding Condition (2)

$$U \in \xi(x) \Rightarrow (\exists V \in \xi(x))\big(y \in V \Rightarrow U \in \xi(y)\big)$$

$$\begin{array}{ccccccc}
x_{11} & x_{12} & x_{13} & \cdots & x_{1n} & \cdots & \to p_1 \\
x_{21} & x_{22} & x_{23} & \cdots & x_{2n} & \cdots & \to p_2 \\
x_{31} & x_{32} & x_{33} & \cdots & x_{3n} & \cdots & \to p_3 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} & \cdots & \to p_m \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
 & & & & & & \downarrow \\
 & & & & & & p
\end{array}$$

Understanding Condition (2)

$$U \in \xi(x) \Rightarrow (\exists V \in \xi(x))\big(y \in V \Rightarrow U \in \xi(y)\big)$$

$$\begin{array}{ccccccc}
x_{11} & x_{12} & x_{13} & \cdots & x_{1n} & \cdots & \to p_1 \\
x_{21} & x_{22} & x_{23} & \cdots & x_{2n} & \cdots & \to p_2 \\
x_{31} & x_{32} & x_{33} & \cdots & x_{3n} & \cdots & \to p_3 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} & \cdots & \to p_m \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
 & & & & & \searrow & \downarrow \\
 & & & & & & p
\end{array}$$

Condition (2) is the filter version of this familiar fact for sequences of numbers
henceforth called
the *sequence condition*

Understanding Condition (2)

$$U \in \xi(x) \Rightarrow (\exists V \in \xi(x))\big(y \in V \Rightarrow U \in \xi(y)\big)$$

Condition (2) is equivalent to:

$$\mathcal{F} \supseteq \xi(x), \mathcal{G}_p \supseteq \xi(p) \Rightarrow \bigcup_{F \in \mathcal{F}} \bigcap_{p \in F} \mathcal{G}_p \supseteq \xi(x) \tag{3}$$

Understanding Condition (2)

$$U \in \xi(x) \Rightarrow (\exists V \in \xi(x))\big(y \in V \Rightarrow U \in \xi(y)\big)$$

Condition (2) is equivalent to:

$$\mathcal{F} \supseteq \xi(x), \mathcal{G}_p \supseteq \xi(p) \Rightarrow \bigcup_{F \in \mathcal{F}} \bigcap_{p \in F} \mathcal{G}_p \supseteq \xi(x) \tag{3}$$

in topological spaces, define $\mathcal{F} \to x$ if $\mathcal{F} \supseteq \xi(x)$

Understanding Condition (2)

$$U \in \xi(x) \Rightarrow (\exists V \in \xi(x))\big(y \in V \Rightarrow U \in \xi(y)\big)$$

Condition (2) is equivalent to:

$$\mathcal{F} \supseteq \xi(x), \mathcal{G}_p \supseteq \xi(p) \Rightarrow \bigcup_{F \in \mathcal{F}} \bigcap_{p \in F} \mathcal{G}_p \supseteq \xi(x) \tag{3}$$

in topological spaces, define $\mathcal{F} \to x$ if $\mathcal{F} \supseteq \xi(x)$

$\{\mathcal{F} : \mathcal{F} \to x\}$ is a filter of filters

Understanding Condition (2)

$$U \in \xi(x) \Rightarrow (\exists V \in \xi(x))\big(y \in V \Rightarrow U \in \xi(y)\big)$$

Condition (2) is equivalent to:

$$\mathcal{F} \supseteq \xi(x), \mathcal{G}_p \supseteq \xi(p) \Rightarrow \bigcup_{F \in \mathcal{F}} \bigcap_{p \in F} \mathcal{G}_p \supseteq \xi(x) \tag{3}$$

in topological spaces, define $\mathcal{F} \to x$ if $\mathcal{F} \supseteq \xi(x)$

$\{\mathcal{F} : \mathcal{F} \to x\}$ is a filter of filters

(3) is now completely in terms of *lattice operations* on $\boldsymbol{Fil}(X)$ ...

Topological Spaces II

Theorem 2.2 ([Kowalsky, 1965, §5])

Given any set X and a function $X \xrightarrow{\Gamma} \boldsymbol{Fil}(\boldsymbol{Fil}(X))$ such that:

$$\dot{x} = \{A \subseteq X : x \in A\} \in \Gamma(x), \tag{4}$$

$$\mathfrak{a} \subseteq \Gamma(x) \Rightarrow \bigcap \mathfrak{a} \in \Gamma(x), \tag{5}$$

and

$$\mathcal{F} \in \Gamma(x), \mathcal{G}_p \in \Gamma(p) \Rightarrow \bigcup_{F \in \mathcal{F}} \bigcap_{p \in F} \mathcal{G}_p \in \Gamma(x) \tag{6}$$

there exists a unique topology Ξ on X such that for each $x \in X$ the set $\gamma(x) = \bigcap \Gamma(x) \in \Gamma(x)$ is the set of all neighbourhoods of x in the topological space (X, Ξ).

[Kowalsky, 1965]

Kowalsky, H. (1965).
General Topology.
Academic Press, New York.

Topological Spaces II

Theorem 2.2 ([Kowalsky, 1965, §5])

Given any set X and a function $X \xrightarrow{\Gamma} \boldsymbol{Fil}(\boldsymbol{Fil}(X))$ such that:

$$\dot{x} = \{A \subseteq X : x \in A\} \in \Gamma(x), \tag{4}$$

$$\mathfrak{a} \subseteq \Gamma(x) \Rightarrow \bigcap \mathfrak{a} \in \Gamma(x), \tag{5}$$

and

$$\mathcal{F} \in \Gamma(x), \mathcal{G}_p \in \Gamma(p) \Rightarrow \bigcup_{F \in \mathcal{F}} \bigcap_{p \in F} \mathcal{G}_p \in \Gamma(x) \tag{6}$$

there exists a unique topology Ξ on X such that for each $x \in X$ the set $\gamma(x) = \bigcap \Gamma(x) \in \Gamma(x)$ is the set of all neighbourhoods of x in the topological space (X, Ξ).

this formulation leads to successive extensions of $\boldsymbol{Top}$ . . .

[Kowalsky, 1965]

Kowalsky, H. (1965).
General Topology.
Academic Press, New York.

Pretopological Spaces

Definition 2.3 (Pretopological Space (see [Bentley et al., 1991], [Herrlich et al., 1991]))

A *pretopological space* is a pair (X, Γ),

Pretopological Spaces

Definition 2.3 (Pretopological Space (see [Bentley et al., 1991], [Herrlich et al., 1991]))

A *pretopological space* is a pair (X, Γ), where X is a set, $X \xrightarrow{\Gamma} \boldsymbol{Fil}(\boldsymbol{Fil}(X))$ is a function such that

Pretopological Spaces

Definition 2.3 (Pretopological Space (see [Bentley et al., 1991], [Herrlich et al., 1991]))

A *pretopological space* is a pair (X, Γ), where X is a set, $X \xrightarrow{\Gamma} \boldsymbol{Fil}(\boldsymbol{Fil}(X))$ is a function such that for each $x \in X$, $\dot{x} \in \Gamma(x)$

Pretopological Spaces

Definition 2.3 (Pretopological Space (see [Bentley et al., 1991], [Herrlich et al., 1991]))

A *pretopological space* is a pair (X, Γ), where X is a set, $X \xrightarrow{\Gamma} \boldsymbol{Fil}(\boldsymbol{Fil}(X))$ is a function such that for each $x \in X$, $\dot{x} \in \Gamma(x)$ and $\Gamma(x)$ is closed under intersections.

[Bentley et al., 1991] [Herrlich et al., 1991]

Bentley, L., Herrlich, H., and Lowen, R. (1991).
Improving constructions in topology.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 3–20. Heldermann, Berlin.

Herrlich, H., Lowen-Colebunders, E., and Schwarz, F. (1991).
Improving $\boldsymbol{Top}$: $\boldsymbol{PrTop}$ and $\boldsymbol{PsTop}$.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 21–34. Heldermann, Berlin.

Pretopological Spaces

For any function $X \xrightarrow{f} Y$ and any $\mathcal{F} \in \boldsymbol{Fil}(X)$:

$$\overrightarrow{f}\mathcal{F} = \{B \subseteq Y : f^{-1}B \in \mathcal{F}\}.$$

Pretopological Spaces

For any function $X \xrightarrow{f} Y$ and any $\mathcal{F} \in \boldsymbol{Fil}(X)$:

$$\overrightarrow{f}\mathcal{F} = \left\{B \subseteq Y : f^{-1}B \in \mathcal{F}\right\}.$$

Clearly, $\overrightarrow{f}\mathcal{F}$ is the filter on Y generated by the images $f(A)$, $A \in \mathcal{F}$.

Pretopological Spaces

Definition 2.3 (Pretopological Morphism (see [Bentley et al., 1991], [Herrlich et al., 1991]))

Given the pretopological spaces (X, Γ) and (Y, Φ),

[Bentley et al., 1991] [Herrlich et al., 1991]

Bentley, L., Herrlich, H., and Lowen, R. (1991).
Improving constructions in topology.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 3–20. Heldermann, Berlin.

Herrlich, H., Lowen-Colebunders, E., and Schwarz, F. (1991).
Improving $\boldsymbol{Top}$: $\boldsymbol{PrTop}$ and $\boldsymbol{PsTop}$.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 21–34. Heldermann, Berlin.

Pretopological Spaces

Definition 2.3 (Pretopological Morphism (see [Bentley et al., 1991], [Herrlich et al., 1991]))

Given the pretopological spaces (X, Γ) and (Y, Φ),a function $X \xrightarrow{f} Y$ is said to be a *pretopological morphism*, if

[Bentley et al., 1991] [Herrlich et al., 1991]

Bentley, L., Herrlich, H., and Lowen, R. (1991).
Improving constructions in topology.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 3–20. Heldermann, Berlin.

Herrlich, H., Lowen-Colebunders, E., and Schwarz, F. (1991).
Improving $\boldsymbol{Top}$: $\boldsymbol{PrTop}$ and $\boldsymbol{PsTop}$.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 21–34. Heldermann, Berlin.

Pretopological Spaces

Definition 2.3 (Pretopological Morphism (see [Bentley et al., 1991], [Herrlich et al., 1991]))

Given the pretopological spaces (X, Γ) and (Y, Φ),a function $X \xrightarrow{f} Y$ is said to be a *pretopological morphism*, if

$$\mathcal{F} \in \Gamma(x) \Rightarrow \overrightarrow{f}\mathcal{F} \in \Phi(f(x)).$$

[Bentley et al., 1991] [Herrlich et al., 1991]

Bentley, L., Herrlich, H., and Lowen, R. (1991).
Improving constructions in topology.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 3–20. Heldermann, Berlin.

Herrlich, H., Lowen-Colebunders, E., and Schwarz, F. (1991).
Improving $\boldsymbol{Top}$: $\boldsymbol{PrTop}$ and $\boldsymbol{PsTop}$.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 21–34. Heldermann, Berlin.

Pretopological Spaces

Definition 2.3 (Pretopological Morphism (see [Bentley et al., 1991], [Herrlich et al., 1991]))

Given the pretopological spaces (X, Γ) and (Y, Φ),a function $X \xrightarrow{f} Y$ is said to be a *pretopological morphism*, if

$$\mathcal{F} \in \Gamma(x) \Rightarrow \vec{f}\mathcal{F} \in \Phi(f(x)).$$

$\boldsymbol{prTop}$ is the category of pretopological spaces and pretopological morphisms.

[Bentley et al., 1991] [Herrlich et al., 1991]

Bentley, L., Herrlich, H., and Lowen, R. (1991).
Improving constructions in topology.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 3–20. Heldermann, Berlin.

Herrlich, H., Lowen-Colebunders, E., and Schwarz, F. (1991).
Improving $\boldsymbol{Top}$: $\boldsymbol{PrTop}$ and $\boldsymbol{PsTop}$.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 21–34. Heldermann, Berlin.

Pseudotopological Spaces

Definition 2.4 (Pseudotopological Space (see [Bentley et al., 1991], [Herrlich et al., 1991]))

A *pseudotopological space* is a pair (X, Γ),

Pseudotopological Spaces

Definition 2.4 (Pseudotopological Space (see [Bentley et al., 1991], [Herrlich et al., 1991]))

A *pseudotopological space* is a pair (X, Γ), where X is a set, $X \xrightarrow{\Gamma} \boldsymbol{Fil}(\boldsymbol{Fil}(X))$ is a function such that

Pseudotopological Spaces

Definition 2.4 (Pseudotopological Space (see [Bentley et al., 1991], [Herrlich et al., 1991]))

A *pseudotopological space* is a pair (X, Γ), where X is a set, $X \xrightarrow{\Gamma} \boldsymbol{Fil}(\boldsymbol{Fil}(X))$ is a function such that for each $x \in X$, $\dot{x} \in \Gamma(x)$ and

Pseudotopological Spaces

Definition 2.4 (Pseudotopological Space (see [Bentley et al., 1991], [Herrlich et al., 1991]))

A *pseudotopological space* is a pair (X, Γ), where X is a set, $X \xrightarrow{\Gamma} \boldsymbol{Fil}(\boldsymbol{Fil}(X))$ is a function such that for each $x \in X$, $\dot{x} \in \Gamma(x)$ and for each filter $\mathcal{F}$ on X, maximal filter $\mathcal{U}$ on X:

$$(\mathcal{F} \subseteq \mathcal{U} \Rightarrow \mathcal{U} \in \Gamma(x)) \Rightarrow \mathcal{F} \in \Gamma(x).$$

[Bentley et al., 1991] [Herrlich et al., 1991]

Bentley, L., Herrlich, H., and Lowen, R. (1991).
Improving constructions in topology.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 3–20. Heldermann, Berlin.

Herrlich, H., Lowen-Colebunders, E., and Schwarz, F. (1991).
Improving $\boldsymbol{Top}$: $\boldsymbol{PrTop}$ and $\boldsymbol{PsTop}$.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 21–34. Heldermann, Berlin.

Pseudotopological Spaces

Definition 2.4 (Pseudotopological Morphism (see [Bentley et al., 1991], [Herrlich et al., 1991]))

Given the pseudotopological spaces (X, Γ) and (Y, Φ),

[Bentley et al., 1991] [Herrlich et al., 1991]

Bentley, L., Herrlich, H., and Lowen, R. (1991).
Improving constructions in topology.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 3–20. Heldermann, Berlin.

Herrlich, H., Lowen-Colebunders, E., and Schwarz, F. (1991).
Improving $\boldsymbol{Top}$: $\boldsymbol{PrTop}$ and $\boldsymbol{PsTop}$.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 21–34. Heldermann, Berlin.

Pseudotopological Spaces

Definition 2.4 (Pseudotopological Morphism (see [Bentley et al., 1991], [Herrlich et al., 1991]))

Given the pseudotopological spaces (X, Γ) and (Y, Φ), a function $X \xrightarrow{f} Y$ is said to be a *pseudotopological morphism*, if

[Bentley et al., 1991] [Herrlich et al., 1991]

Bentley, L., Herrlich, H., and Lowen, R. (1991).
Improving constructions in topology.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 3–20. Heldermann, Berlin.

Herrlich, H., Lowen-Colebunders, E., and Schwarz, F. (1991).
Improving $\boldsymbol{Top}$: $\boldsymbol{PrTop}$ and $\boldsymbol{PsTop}$.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 21–34. Heldermann, Berlin.

Pseudotopological Spaces

Definition 2.4 (Pseudotopological Morphism (see [Bentley et al., 1991], [Herrlich et al., 1991]))

Given the pseudotopological spaces (X,Γ) and (Y,Φ), a function $X \xrightarrow{f} Y$ is said to be a *pseudotopological morphism*, if

$$\mathcal{F} \in \Gamma(x) \Rightarrow \overrightarrow{f}\mathcal{F} \in \Phi(f(x)).$$

[Bentley et al., 1991] [Herrlich et al., 1991]

Bentley, L., Herrlich, H., and Lowen, R. (1991).
Improving constructions in topology.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 3–20. Heldermann, Berlin.

Herrlich, H., Lowen-Colebunders, E., and Schwarz, F. (1991).
Improving $\boldsymbol{Top}$: $\boldsymbol{PrTop}$ and $\boldsymbol{PsTop}$.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 21–34. Heldermann, Berlin.

Pseudotopological Spaces

Definition 2.4 (Pseudotopological Morphism (see [Bentley et al., 1991], [Herrlich et al., 1991]))

Given the pseudotopological spaces (X, Γ) and (Y, Φ), a function $X \xrightarrow{f} Y$ is said to be a *pseudotopological morphism*, if

$$\mathcal{F} \in \Gamma(x) \Rightarrow \overrightarrow{f}\mathcal{F} \in \Phi(f(x)).$$

$\boldsymbol{psTop}$ is the category of pseudotopological spaces and pseudotopological morphisms.

[Bentley et al., 1991] [Herrlich et al., 1991]

Bentley, L., Herrlich, H., and Lowen, R. (1991).
Improving constructions in topology.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 3–20. Heldermann, Berlin.

Herrlich, H., Lowen-Colebunders, E., and Schwarz, F. (1991).
Improving $\boldsymbol{Top}$: $\boldsymbol{PrTop}$ and $\boldsymbol{PsTop}$.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 21–34. Heldermann, Berlin.

Key Issues for Generalisation

A. main focus is on the lattice of subobjects

Key Issues for Generalisation

A. main focus is on the lattice of subobjects

B. one needs to avoid the atomicity present for sets

Key Issues for Generalisation

A. main focus is on the lattice of subobjects

B. one needs to avoid the atomicity present for sets

C. filters can be defined on any semilattice

Key Issues for Generalisation

A. main focus is on the lattice of subobjects

B. one needs to avoid the atomicity present for sets

C. filters can be defined on any semilattice

D. known from [Iberkleid and McGovern, 2009]: a lattice L is distributive, if and only if, $\boldsymbol{Fil}(L)$ is a coherent frame

[Iberkleid and McGovern, 2009]

Iberkleid, W. and McGovern, W. W. (2009).
A Natural Equivalence for the Category of Coherent Frames.
Alg. Univ., 62:247–258.
available at: http://home.fau.edu/wmcgove1/web/Papers/WolfNatEq.pdf.

Key Issues for Generalisation

A. main focus is on the lattice of subobjects

B. one needs to avoid the atomicity present for sets

C. filters can be defined on any semilattice

D. known from [Iberkleid and McGovern, 2009]: a lattice L is distributive, if and only if, $\boldsymbol{Fil}(L)$ is a coherent frame
A frame is *coherent* if it is compact, algebraic and binary meets of compact elements is compact.

Key Issues for Generalisation

A. main focus is on the lattice of subobjects

B. one needs to avoid the atomicity present for sets

C. filters can be defined on any semilattice

D. known from [Iberkleid and McGovern, 2009]: a lattice L is distributive, if and only if, $\boldsymbol{Fil}(L)$ is a coherent frame
A frame is *coherent* if it is compact, algebraic and binary meets of compact elements is compact.

E. one needs a *nice* factorisation of morphisms

Our Setup

1. $\mathbb{A}$ is a finitely complete category

Our Setup

Definition 3.1 (Proper Factorisations)

A *proper factorisation system* for $\mathbb{A}$ is $(\boldsymbol{E}, \boldsymbol{M})$, where:

Our Setup

Definition 3.1 (Proper Factorisations)

A *proper factorisation system* for $\mathbb{A}$ is $(\boldsymbol{E}, \boldsymbol{M})$, where:

(a) $\boldsymbol{E}$ is a set of epimorphisms of $\mathbb{A}$,

Our Setup

Definition 3.1 (Proper Factorisations)

A *proper factorisation system* for $\mathbb{A}$ is $(\boldsymbol{E}, \boldsymbol{M})$, where:

(a) $\boldsymbol{E}$ is a set of epimorphisms of $\mathbb{A}$,

(b) $\boldsymbol{M}$ is a set of monomorphisms of $\mathbb{A}$,

Our Setup

Definition 3.1 (Proper Factorisations)

A *proper factorisation system* for $\mathbb{A}$ is $(\boldsymbol{E}, \boldsymbol{M})$, where:

(a) $\boldsymbol{E}$ is a set of epimorphisms of $\mathbb{A}$,

(b) $\boldsymbol{M}$ is a set of monomorphisms of $\mathbb{A}$,

(c) for every morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$,

Our Setup

Definition 3.1 (Proper Factorisations)

A *proper factorisation system* for $\mathbb{A}$ is $(\boldsymbol{E}, \boldsymbol{M})$, where:

(a) $\boldsymbol{E}$ is a set of epimorphisms of $\mathbb{A}$,

(b) $\boldsymbol{M}$ is a set of monomorphisms of $\mathbb{A}$,

(c) for every morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$, there exists a $X \xrightarrow{e} I$ in $\boldsymbol{E}$

Our Setup

Definition 3.1 (Proper Factorisations)

A *proper factorisation system* for $\mathbb{A}$ is $(\boldsymbol{E}, \boldsymbol{M})$, where:

(a) $\boldsymbol{E}$ is a set of epimorphisms of $\mathbb{A}$,

(b) $\boldsymbol{M}$ is a set of monomorphisms of $\mathbb{A}$,

(c) for every morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$, there exists a $X \xrightarrow{e} I$ in $\boldsymbol{E}$ and a $I \xrightarrow{m} Y$ in $\boldsymbol{M}$ such that

Our Setup

Definition 3.1 (Proper Factorisations)

A *proper factorisation system* for $\mathbb{A}$ is $(\boldsymbol{E}, \boldsymbol{M})$, where:

(a) $\boldsymbol{E}$ is a set of epimorphisms of $\mathbb{A}$,

(b) $\boldsymbol{M}$ is a set of monomorphisms of $\mathbb{A}$,

(c) for every morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$, there exists a $X \xrightarrow{e} I$ in $\boldsymbol{E}$ and a $I \xrightarrow{m} Y$ in $\boldsymbol{M}$ such that
$$\begin{tikzcd} X \arrow[r, "e"] \arrow[rr, bend right, "f"'] & I \arrow[r, "m"] & Y \end{tikzcd},$$
and

Our Setup

Definition 3.1 (Proper Factorisations)

A *proper factorisation system* for $\mathbb{A}$ is $(\boldsymbol{E}, \boldsymbol{M})$, where:

(a) $\boldsymbol{E}$ is a set of epimorphisms of $\mathbb{A}$,

(b) $\boldsymbol{M}$ is a set of monomorphisms of $\mathbb{A}$,

(c) for every morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$, there exists a $X \xrightarrow{e} I$ in $\boldsymbol{E}$ and a $I \xrightarrow{m} Y$ in $\boldsymbol{M}$ such that $\begin{tikzcd} X \arrow[r, "e"] \arrow[rr, bend right, "f"'] & I \arrow[r, "m"] & Y \end{tikzcd}$, and

(d) if $v \circ e = m \circ u$ for some $e \in \boldsymbol{E}$ and $m \in \boldsymbol{M}$

Our Setup

Definition 3.1 (Proper Factorisations)

A *proper factorisation system* for $\mathbb{A}$ is $(\boldsymbol{E}, \boldsymbol{M})$, where:

(a) $\boldsymbol{E}$ is a set of epimorphisms of $\mathbb{A}$,

(b) $\boldsymbol{M}$ is a set of monomorphisms of $\mathbb{A}$,

(c) for every morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$, there exists a $X \xrightarrow{e} I$ in $\boldsymbol{E}$ and a $I \xrightarrow{m} Y$ in $\boldsymbol{M}$ such that
$$\begin{tikzcd} X \arrow[r, "e"] \arrow[rr, bend right, "f"'] & I \arrow[r, "m"] & Y \end{tikzcd},$$
and

(d) if $v \circ e = m \circ u$ for some $e \in \boldsymbol{E}$ and $m \in \boldsymbol{M}$ then there exists a unique morphism w such that

Our Setup

Definition 3.1 (Proper Factorisations)

A *proper factorisation system* for $\mathbb{A}$ is $(\boldsymbol{E}, \boldsymbol{M})$, where:

(a) $\boldsymbol{E}$ is a set of epimorphisms of $\mathbb{A}$,

(b) $\boldsymbol{M}$ is a set of monomorphisms of $\mathbb{A}$,

(c) for every morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$, there exists a $X \xrightarrow{e} I$ in $\boldsymbol{E}$ and a $I \xrightarrow{m} Y$ in $\boldsymbol{M}$ such that
$$\begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[rr, bend right, "f"'] & I \arrow[r, tail, "m"] & Y \end{tikzcd},$$
and

(d) if $v \circ e = m \circ u$ for some $e \in \boldsymbol{E}$ and $m \in \boldsymbol{M}$ then there exists a unique morphism w such that
$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "u"'] & \cdot \arrow[d, "v"] \arrow[dl, dotted, "!\, w"] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$$
commutes.

Our Setup

1. $\mathbb{A}$ is a finitely complete category
2. $\mathbb{A}$ has a proper factorisation system $(\boldsymbol{E}, \boldsymbol{M})$ (see [Carboni et al., 1997, §2] for details on factorisation systems)

[Carboni et al., 1997]

Carboni, A., Janelidze, G., Kelly, G. M., and Pare, R. (1997).
On localization and stabilization for factorization systems.
Appl. Categ. Str., 5(1):1 – 58.

Our Setup

Definition 3.1 (Admissible Subobjects)

Any $M \xrightarrow{m} X$ ($m \in \boldsymbol{M}$) is an *admissible subobject* of X.

Our Setup

Let $\boldsymbol{Sub}_M(X)$ be the set of admissible subobjects.

Our Setup

Let $\boldsymbol{Sub_M}(X)$ be the set of admissible subobjects.

(a) For $m, n \in \boldsymbol{Sub_M}(X)$, $m \leq n$ if there exists a morphism k making the diagram:

$$\begin{tikzcd} M \arrow[r, dotted, "!\,k"] \arrow[dr, "m"'] & N \arrow[d, "n"] \\ & X \end{tikzcd}$$

to commute, i.e., $m = n \circ k$.

Our Setup

Let $\boldsymbol{Sub}_M(X)$ be the set of admissible subobjects.

(a) For $m, n \in \boldsymbol{Sub}_M(X)$, $m \leq n$ if there exists a morphism k making the diagram:

$$\begin{tikzcd} M \arrow[r, dotted, "!\,k"] \arrow[dr, "m"'] & N \arrow[d, "n"] \\ & X \end{tikzcd}$$

to commute, i.e., $m = n \circ k$.

(b) In presence of finite completeness of $\mathbb{A}$, $\boldsymbol{Sub}_M(X)$ is a meet semilattice.

Our Setup

1. $\mathbb{A}$ is a finitely complete category
2. $\mathbb{A}$ has a proper factorisation system $(\boldsymbol{E}, \boldsymbol{M})$ (see [Carboni et al., 1997, §2] for details on factorisation systems)
3. Each $\boldsymbol{Sub_M}(X)$ is a distributive complete lattice

Our Setup

Example 3.1 (Examples of Setup)

(a) Every Grothendieck topos

Our Setup

Example 3.1 (Examples of Setup)

(a) Every Grothendieck topos

(b) Every small complete, small cocomplete and extensive category (see [Carboni et al., 1993] for extensive and distributive categories)

[Carboni et al., 1997]

Carboni, A., Janelidze, G., Kelly, G. M., and Pare, R. (1997).
On localization and stabilization for factorization systems.
Appl. Categ. Str., 5(1):1 – 58.

[Carboni et al., 1993]

Carboni, A., Lack, S., and Walters, R. (1993).
Introduction to Extensive and Distributive Categories.
Jr. Pure Appl. Alg., 84:145 – 158.

Our Setup

Example 3.1 (Examples of Setup)

(a) Every Grothendieck topos

(b) Every small complete, small cocomplete and extensive category (see [Carboni et al., 1993] for extensive and distributive categories)

(c) If $\mathbb{A}$ be such and A be an object, then so also is $(\mathbb{A} \downarrow A)$ (see [Clementino et al., 2004, §2.10])

[Carboni et al., 1997]

Carboni, A., Janelidze, G., Kelly, G. M., and Pare, R. (1997).
On localization and stabilization for factorization systems.
Appl. Categ. Str., 5(1):1 – 58.

[Carboni et al., 1993]

Carboni, A., Lack, S., and Walters, R. (1993).
Introduction to Extensive and Distributive Categories.
Jr. Pure Appl. Alg., 84:145 – 158.

[Clementino et al., 2004]

Clementino, M. M., Giuli, E., and Tholen, W. (2004).
A Functional Approach to General Topology.
In *Categorical Foundations*, number 97 in Encyclopedia Math. Appl., pages 103–163. Cambridge University Press.

Our Setup

Example 3.1 (Examples of Setup)

(a) Every Grothendieck topos

(b) Every small complete, small cocomplete and extensive category (see [Carboni et al., 1993] for extensive and distributive categories)

(c) If $\mathbb{A}$ be such and A be an object, then so also is $(\mathbb{A} \downarrow A)$ (see [Clementino et al., 2004, §2.10])

(d) Hence many of the examples of [Clementino et al., 2004] is an example of this context

[Carboni et al., 1997]

Carboni, A., Janelidze, G., Kelly, G. M., and Pare, R. (1997).
On localization and stabilization for factorization systems.
Appl. Categ. Str., 5(1):1 – 58.

[Carboni et al., 1993]

Carboni, A., Lack, S., and Walters, R. (1993).
Introduction to Extensive and Distributive Categories.
Jr. Pure Appl. Alg., 84:145 – 158.

[Clementino et al., 2004]

Clementino, M. M., Giuli, E., and Tholen, W. (2004).
A Functional Approach to General Topology.
In *Categorical Foundations*, number 97 in Encyclopedia Math. Appl., pages 103–163. Cambridge University Press.

Our Setup

1. $\mathbb{A}$ is a finitely complete category
2. $\mathbb{A}$ has a proper factorisation system $(\boldsymbol{E}, \boldsymbol{M})$ (see [Carboni et al., 1997, §2] for details on factorisation systems)
3. Each $\boldsymbol{Sub}_M(X)$ is a distributive complete lattice

Neighbourhoods of all Sorts

Definition 3.1 (Pre- & Weak Neighbourhoods on an Object)

A *pre-neighbourhood* on an object X is an order preserving map

Neighbourhoods of all Sorts

Definition 3.1 (Pre- & Weak Neighbourhoods on an Object)

A *pre-neighbourhood* on an object X is an order preserving map $\boldsymbol{Sub_M}(X)^{\mathrm{op}} \xrightarrow{\mu} \boldsymbol{Fil}(X)$ such that:

Neighbourhoods of all Sorts

Definition 3.1 (Pre- & Weak Neighbourhoods on an Object)

A *pre-neighbourhood* on an object X is an order preserving map $\boldsymbol{Sub_M}(X)^{\mathrm{op}} \xrightarrow{\mu} \boldsymbol{Fil}(X)$ such that:

$$n \in \mu(m) \Rightarrow m \leq n.$$

Neighbourhoods of all Sorts

Definition 3.1 (Pre- & Weak Neighbourhoods on an Object)

A *pre-neighbourhood* on an object X is an order preserving map $\boldsymbol{Sub_M}(X)^{\mathrm{op}} \xrightarrow{\mu} \boldsymbol{Fil}(X)$ such that:

$$n \in \mu(m) \Rightarrow m \leq n.$$

A pre-neighbourhood μ on X such that:

$$\mu(m) = \bigcup_{p \in \mu(m)} \mu(p)$$

Neighbourhoods of all Sorts

Definition 3.1 (Pre- & Weak Neighbourhoods on an Object)

A *pre-neighbourhood* on an object X is an order preserving map $\boldsymbol{Sub_M}(X)^{\mathrm{op}} \xrightarrow{\mu} \boldsymbol{Fil}(X)$ such that:

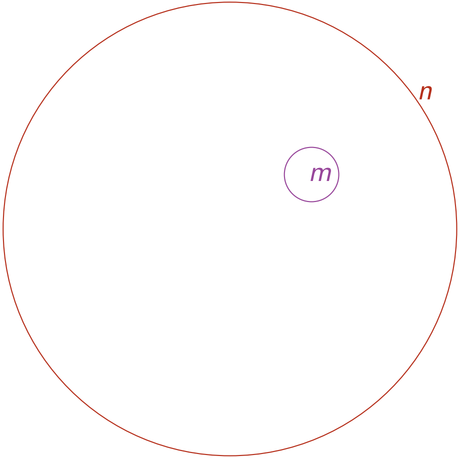
$$n \in \mu(m) \Rightarrow m \leq n.$$

A pre-neighbourhood μ on X such that:

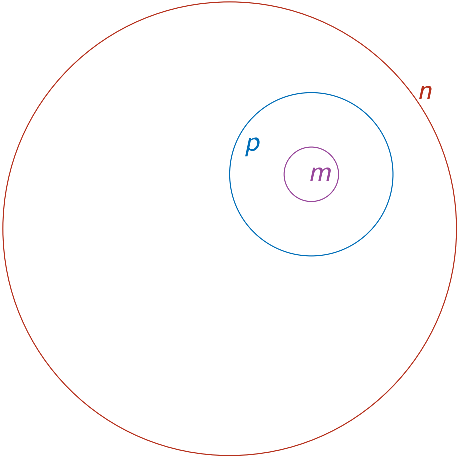
$$\mu(m) = \bigcup_{p \in \mu(m)} \mu(p)$$

is a *weak neighbourhood* on X.

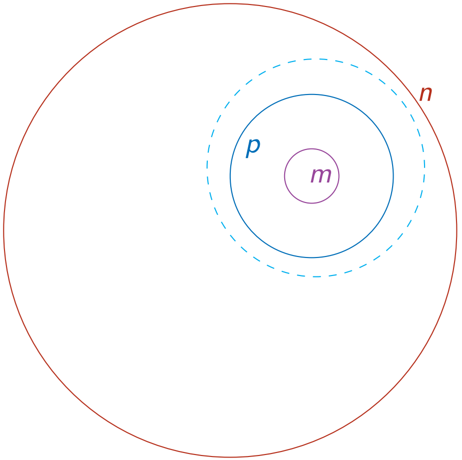
Neighbourhoods of all Sorts



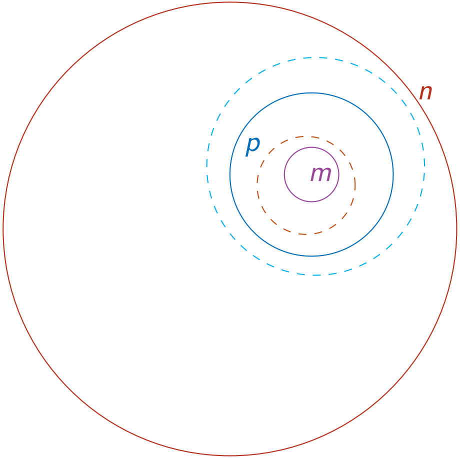
Neighbourhoods of all Sorts



Neighbourhoods of all Sorts



Neighbourhoods of all Sorts



weak neighbourhoods are precisely the *interpolative pre-neighbourhoods*

Neighbourhoods of all Sorts

Definition 3.1 (Neighbourhoods on an Object)

A weak neighbourhood μ on X such that:

Neighbourhoods of all Sorts

Definition 3.1 (Neighbourhoods on an Object)

A weak neighbourhood μ on X such that:

$$\mu(\bigvee G) = \bigcap_{g \in G} \mu(g), \quad \text{for all } G \subseteq \boldsymbol{Sub}_M(X)$$

Neighbourhoods of all Sorts

Definition 3.1 (Neighbourhoods on an Object)

A weak neighbourhood μ on X such that:

$$\mu(\bigvee G) = \bigcap_{g \in G} \mu(g), \qquad \text{for all } G \subseteq \boldsymbol{Sub}_M(X)$$

is a *neighbourhood* on X.

Neighbourhoods of all Sorts

Definition 3.1 (Neighbourhoods on an Object)

A weak neighbourhood μ on X such that:

$$\mu(\bigvee G) = \bigcap_{g \in G} \mu(g), \qquad \text{for all } G \subseteq \boldsymbol{Sub_M}(X)$$

is a *neighbourhood* on X.

Neighbourhoods are meet preserving weak neighbourhoods.

Interior and *Open*

Theorem 3.2 (μ-Interior and μ-Open)

Given a pre-neighbourhood μ on X let:

Interior and *Open*

Theorem 3.2 (μ-Interior and μ-Open)

Given a pre-neighbourhood μ on X let:

$$\mathfrak{O}_\mu = \{p \in \boldsymbol{Sub}_M(X) : p \in \mu(p)\}, \tag{7}$$

and

$$\mathrm{int}_\mu(m) = \bigvee\{p \in \mathfrak{O}_\mu : p \leq m\}. \tag{8}$$

Interior and *Open*

Theorem 3.2 (μ-Interior and μ-Open)

Given a pre-neighbourhood μ on X let:

$$\mathfrak{O}_\mu = \{p \in \boldsymbol{Sub_M}(X) : p \in \mu(p)\}, \tag{7}$$

and

$$\mathrm{int}_\mu(m) = \bigvee \{p \in \mathfrak{O}_\mu : p \leq m\}. \tag{8}$$

Then:

(a) $\mathfrak{O}_\mu$ is closed under finite meets.

Interior and *Open*

Theorem 3.2 (μ-Interior and μ-Open)

Given a pre-neighbourhood μ on X let:

$$\mathfrak{O}_\mu = \{p \in \boldsymbol{Sub_M}(X) : p \in \mu(p)\}, \tag{7}$$

and

$$\mathrm{int}_\mu(m) = \bigvee\{p \in \mathfrak{O}_\mu : p \leq m\}. \tag{8}$$

Then:

(a) $\mathfrak{O}_\mu$ is closed under finite meets.

(b) $\mathfrak{O}_\mu$ is closed under arbitrary joins, if and only if, μ preserve meets.

Interior and *Open*

Theorem 3.2 (μ-Interior and μ-Open)

Given a pre-neighbourhood μ on X let:

$$\mathfrak{O}_\mu = \{p \in \boldsymbol{Sub_M}(X) : p \in \mu(p)\}, \tag{7}$$

and

$$\mathrm{int}_\mu(m) = \bigvee\{p \in \mathfrak{O}_\mu : p \leq m\}. \tag{8}$$

Then:

(a) $\mathfrak{O}_\mu$ *is closed under finite meets.*

(b) $\mathfrak{O}_\mu$ *is closed under arbitrary joins, if and only if, μ preserve meets.*

(c) *If μ is a neighbourhood on X then $m \mapsto \mathrm{int}_\mu(m)$ is a Kuratowski interior operator and:*

$$p \in \mu(m) \Leftrightarrow m \leq \mathrm{int}_\mu(p).$$

Interior and *Open*

weak neighbourhoods are not neighbourhoods...

Interior and *Open*

Example 3.2 (Weak neighbourhood **not** a neighbourhood)

Interior and *Open*

Example 3.2 (Weak neighbourhood **not** a neighbourhood)

Let $X = [0, 2]$ and $a_0 = 0$, $a_n = 1 - \frac{1}{n}$, $n \geq 1$.

Interior and *Open*

Example 3.2 (Weak neighbourhood **not** a neighbourhood)

Let $X = [0, 2]$ and $a_0 = 0,\ a_n = 1 - \frac{1}{n},\ n \geq 1.$
For $A \subseteq X$, if $\sup A < 1$ let n_A be the smallest $n \geq 1$ such that:

$$x \in A \Rightarrow x \leq 1 - \frac{1}{n_A}.$$

Interior and *Open*

Example 3.2 (Weak neighbourhood **not** a neighbourhood)

Let $X = [0, 2]$ and $a_0 = 0,\ a_n = 1 - \frac{1}{n},\ n \geq 1.$
For $A \subseteq X$, if $\sup A < 1$ let n_A be the smallest $n \geq 1$ such that:

$$x \in A \Rightarrow x \leq 1 - \frac{1}{n_A}.$$

Put:

$$\mu(A) = \begin{cases} \{X\}, & \text{if } A \cap [1,2] \neq \emptyset \text{ or } \sup A = 1 \\ \{T \subseteq X : [0, 1 - \frac{1}{n_A}] \subseteq T\}, & \text{otherwise} \end{cases}$$

Interior and *Open*

Example 3.2 (Weak neighbourhood **not** a neighbourhood)

Let $X = [0, 2]$ and $a_0 = 0$, $a_n = 1 - \frac{1}{n}$, $n \geq 1$.
For $A \subseteq X$, if $\sup A < 1$ let n_A be the smallest $n \geq 1$ such that:

$$x \in A \Rightarrow x \leq 1 - \frac{1}{n_A}.$$

Put:

$$\mu(A) = \begin{cases} \{X\}, & \text{if } A \cap [1, 2] \neq \emptyset \text{ or } \sup A = 1 \\ \{T \subseteq X : [0, 1 - \frac{1}{n_A}] \subseteq T\}, & \text{otherwise} \end{cases}$$

Clearly μ is a pre-neighbourhood on X.

Interior and *Open*

Example 3.2 (Weak neighbourhood **not** a neighbourhood)

Let $X = [0,2]$ and $a_0 = 0,\ a_n = 1 - \frac{1}{n},\ n \geq 1$.
For $A \subseteq X$, if $\sup A < 1$ let n_A be the smallest $n \geq 1$ such that:

$$x \in A \Rightarrow x \leq 1 - \frac{1}{n_A}.$$

Put:

$$\mu(A) = \begin{cases} \{X\}, & \text{if } A \cap [1,2] \neq \emptyset \text{ or } \sup A = 1 \\ \{T \subseteq X : [0, 1 - \frac{1}{n_A}] \subseteq T\}, & \text{otherwise} \end{cases}$$

Further:

$$\mathfrak{O}_\mu = \left\{ \left[0, 1 - \frac{1}{n}\right] : n \geq 1 \right\} \cup \{X\}.$$

Interior and *Open*

Example 3.2 (Weak neighbourhood **not** a neighbourhood)

Let $X = [0,2]$ and $a_0 = 0$, $a_n = 1 - \frac{1}{n}$, $n \geq 1$.
For $A \subseteq X$, if $\sup A < 1$ let n_A be the smallest $n \geq 1$ such that:

$$x \in A \Rightarrow x \leq 1 - \frac{1}{n_A}.$$

Put:

$$\mu(A) = \begin{cases} \{X\}, & \text{if } A \cap [1,2] \neq \emptyset \text{ or } \sup A = 1 \\ \{T \subseteq X : [0, 1 - \frac{1}{n_A}] \subseteq T\}, & \text{otherwise} \end{cases}$$

Further:

$$\mathfrak{O}_\mu = \left\{ \left[0, 1 - \frac{1}{n}\right] : n \geq 1 \right\} \cup \{X\}.$$

Hence: μ is a weak neighbourhood.

Interior and *Open*

Example 3.2 (Weak neighbourhood **not** a neighbourhood)

Let $X = [0, 2]$ and $a_0 = 0$, $a_n = 1 - \frac{1}{n}$, $n \geq 1$.
For $A \subseteq X$, if $\sup A < 1$ let n_A be the smallest $n \geq 1$ such that:

$$x \in A \Rightarrow x \leq 1 - \frac{1}{n_A}.$$

Put:

$$\mu(A) = \begin{cases} \{X\}, & \text{if } A \cap [1,2] \neq \emptyset \text{ or } \sup A = 1 \\ \{T \subseteq X : [0, 1 - \frac{1}{n_A}] \subseteq T\}, & \text{otherwise} \end{cases}$$

Further:

$$\mathfrak{O}_\mu = \left\{ [0, 1 - \frac{1}{n}] : n \geq 1 \right\} \cup \{X\}.$$

But:

$$\{X\} = \mu([0,1)) = \mu(\bigcup_{n \geq 1} [0, 1 - \frac{1}{n}]) \subset \bigcap_{n \geq 1} \mu([0, 1 - \frac{1}{n}]),$$

Interior and *Open*

Example 3.2 (Weak neighbourhood **not** a neighbourhood)

Let $X = [0,2]$ and $a_0 = 0$, $a_n = 1 - \frac{1}{n}$, $n \geq 1$.
For $A \subseteq X$, if $\sup A < 1$ let n_A be the smallest $n \geq 1$ such that:

$$x \in A \Rightarrow x \leq 1 - \frac{1}{n_A}.$$

Put:

$$\mu(A) = \begin{cases} \{X\}, & \text{if } A \cap [1,2] \neq \emptyset \text{ or } \sup A = 1 \\ \{T \subseteq X : [0, 1 - \frac{1}{n_A}] \subseteq T\}, & \text{otherwise} \end{cases}$$

Further:

$$\mathfrak{O}_\mu = \left\{ \left[0, 1 - \frac{1}{n}\right] : n \geq 1 \right\} \cup \{X\}.$$

Hence μ is not a neighbourhood.

Interior and *Open*

a pre-neighbourhood μ such that:

Interior and *Open*

a pre-neighbourhood μ such that:

$$p \in \mu(m) \Leftrightarrow m \leq \mathrm{int}_\mu(p)$$

Interior and *Open*

a pre-neighbourhood μ such that:

$$p \in \mu(m) \Leftrightarrow m \leq \mathrm{int}_{\mu}(p)$$

is a weak neighbourhood, but may not be a neighbourhood!

Interior and *Open*

Example 3.2 (A weak neighbourhood with *interior property* **not** a neighbourhood)

Interior and *Open*

Example 3.2 (A weak neighbourhood with *interior property* **not** a neighbourhood)

Let $(X, \mathcal{T})$ be a topological space with $\mathcal{C}$ its set of closed sets.

Interior and *Open*

Example 3.2 (A weak neighbourhood with *interior property* **not** a neighbourhood)

Let $(X, \mathcal{T})$ be a topological space with $\mathcal{C}$ its set of closed sets.
Define:

$$\mu(A) = \{V \subseteq X : (\exists C \in \mathcal{C})(A \subseteq C \subseteq V)\}.$$

Interior and *Open*

Example 3.2 (A weak neighbourhood with *interior property* **not** a neighbourhood)

Let $(X, \mathcal{T})$ be a topological space with $\mathcal{C}$ its set of closed sets.
Define:

$$\mu(A) = \{V \subseteq X : (\exists C \in \mathcal{C})(A \subseteq C \subseteq V)\}.$$

μ is a pre-neighbourhood.

Interior and *Open*

Example 3.2 (A weak neighbourhood with *interior property* **not** a neighbourhood)

Let $(X, \mathcal{T})$ be a topological space with $\mathcal{C}$ its set of closed sets.
Define:

$$\mu(A) = \{V \subseteq X : (\exists C \in \mathcal{C})(A \subseteq C \subseteq V)\}.$$

$\mathfrak{O}_\mu = \mathcal{C}$.

Interior and *Open*

Example 3.2 (A weak neighbourhood with *interior property* **not** a neighbourhood)

Let $(X, \mathcal{T})$ be a topological space with $\mathcal{C}$ its set of closed sets.
Define:

$$\mu(A) = \{V \subseteq X : (\exists C \in \mathcal{C})(A \subseteq C \subseteq V)\}.$$

$\mathfrak{O}_\mu = \mathcal{C}$.
Hence μ satisfies the *interior property*.

Interior and *Open*

Example 3.2 (A weak neighbourhood with *interior property* **not** a neighbourhood)

Let $(X, \mathcal{T})$ be a topological space with $\mathcal{C}$ its set of closed sets.
Define:

$$\mu(A) = \{V \subseteq X : (\exists C \in \mathcal{C})(A \subseteq C \subseteq V)\}.$$

If $\mathcal{C}$ be not closed under arbitrary joins then μ is not a neighbourhood.

Morphisms

Given any morphism $X \xrightarrow{f} Y$ one obtains:

Morphisms

Given any morphism $X \xrightarrow{f} Y$ one obtains:

(a) from the $(\boldsymbol{E}, \boldsymbol{M})$ factorisation the *image-preimage* adjunction:

Morphisms

Given any morphism $X \xrightarrow{f} Y$ one obtains:

(a) from the $(\boldsymbol{E}, \boldsymbol{M})$ factorisation the *image-preimage* adjunction:

$$\begin{tikzcd} \boldsymbol{Sub_M}(X) \arrow[rr, bend left, "\exists_f"] \arrow[rr, phantom, "\perp"] & & \boldsymbol{Sub_M}(Y) \arrow[ll, bend left, "f^{-1}"] \end{tikzcd}$$

Morphisms

Given any morphism $X \xrightarrow{f} Y$ one obtains:

(a) given $m \in \boldsymbol{Sub}_M(X)$, $y \in \boldsymbol{Sub}_M(Y)$:

Morphisms

Given any morphism $X \xrightarrow{f} Y$ one obtains:

(a) given $m \in \boldsymbol{Sub}_M(X)$, $y \in \boldsymbol{Sub}_M(Y)$:

$$\begin{tikzcd}
f^{-1}N \arrow[r, "f_n"] \arrow[d, "f^{-1}n"'] & N \arrow[d, "n"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

pullback along f defines $n \mapsto f^{-1}n$

Morphisms

Given any morphism $X \xrightarrow{f} Y$ one obtains:

(a) given $m \in \boldsymbol{Sub_M}(X)$, $y \in \boldsymbol{Sub_M}(Y)$:

$$\begin{tikzcd} M \arrow[r, "f\big|_M"] \arrow[d, "m"'] & \exists_f M \arrow[d, "\exists_f m"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

the $(\boldsymbol{E}, \boldsymbol{M})$ factorisation of $f \circ m$ produces $m \mapsto \exists_f m$

Morphisms

Given any morphism $X \xrightarrow{f} Y$ one obtains:

(b) since $\exists_f$ and f^{-1} are both order preserving maps, one has the further adjunction:

Morphisms

Given any morphism $X \xrightarrow{f} Y$ one obtains:

(b) since $\exists_f$ and f^{-1} are both order preserving maps, one has the further adjunction:

$$\begin{tikzcd} \boldsymbol{Fil}(X) \arrow[rr, bend right, "\overrightarrow{f}"'] \arrow[rr, phantom, "\perp"] & & \boldsymbol{Fil}(Y) \arrow[ll, bend right, "\overleftarrow{f}"'] \end{tikzcd},$$

Morphisms

Given any morphism $X \xrightarrow{f} Y$ one obtains:

(b) where:

$$\overrightarrow{f} A = \{ y \in \boldsymbol{Sub}_M(Y) : f^{-1} y \in A \}, \quad \text{for } A \in \boldsymbol{Fil}(X) \tag{7}$$

and

$$\overleftarrow{f} B = \{ x \in \boldsymbol{Sub}_M(X) : (\exists b \in B)(f^{-1} b \leq x) \}, \quad \text{for } B \in \boldsymbol{Fil}(Y), \tag{8}$$

Clearly: $\overrightarrow{f} A$ is the filter generated by the *images* $\exists_f a$ $(a \in A)$, and $\overleftarrow{f} B$ is the filter generated by the *preimages* $f^{-1} b$ $(b \in B)$.

Morphisms

Definition 3.3 (Pre-neighbourhood Morphisms)

Given the pre-neighbourhoods μ on X and ϕ on Y, a morphism $X \xrightarrow{f} Y$ is a *pre-neighbourhood morphism* if:

Morphisms

Definition 3.3 (Pre-neighbourhood Morphisms)

Given the pre-neighbourhoods μ on X and ϕ on Y, a morphism $X \xrightarrow{f} Y$ is a *pre-neighbourhood morphism* if:

$$p \in \phi(n) \Rightarrow f^{-1}p \in \mu(f^{-1}n). \tag{7}$$

Morphisms

Theorem 3.3 ([Holgate and Slapal, 2011, §3])

Given pre-neighbourhood structures μ on X and ν on Y, THE FOLLOWING ARE EQUIVALENT *:*

[Holgate and Slapal, 2011]

Holgate, D. and Slapal, J. (2011).
Categorical neighborhood operators.
Top. Appl., 158:2356–2365.

Morphisms

Theorem 3.3 ([Holgate and Slapal, 2011, §3])

Given pre-neighbourhood structures μ on X and ν on Y, THE FOLLOWING ARE EQUIVALENT *:*

(a) *for each* $n \in \boldsymbol{Sub}_{M}(Y)$, $p \in \phi(n) \Rightarrow f^{-1}p \in \mu(f^{-1}n)$

(b) *for each* $n \in \boldsymbol{Sub}_{M}(Y)$, $\overleftarrow{f}\phi(n) \subseteq \mu(f^{-1}n)$

(c) *for each* $n \in \boldsymbol{Sub}_{M}(Y)$, $\phi(n) \subseteq \overrightarrow{f}\mu(f^{-1}n)$

(d) *for each* $m \in \boldsymbol{Sub}_{M}(X)$, $\overleftarrow{f}\phi(\exists_f m) \leq \mu(m)$

[Holgate and Slapal, 2011]

Holgate, D. and Slapal, J. (2011).
Categorical neighborhood operators.
Top. Appl., 158:2356–2365.

Morphisms

Theorem 3.3 ([Holgate and Slapal, 2011, §3])

given the order preserving maps:

[Holgate and Slapal, 2011]

Holgate, D. and Slapal, J. (2011).
Categorical neighborhood operators.
Top. Appl., 158:2356–2365.

Morphisms

Theorem 3.3 ([Holgate and Slapal, 2011, §3])

given the order preserving maps:

$$\begin{tikzcd}
\boldsymbol{Sub}_{\boldsymbol{M}}(X)^{\mathrm{op}} \arrow[r, shift right, "\exists_f"'] \arrow[r, phantom, "\perp"] \arrow[d, "\mu"] & \boldsymbol{Sub}_{\boldsymbol{M}}(Y)^{\mathrm{op}} \arrow[l, shift right, "f^{-1}"'] \arrow[d, "\phi"] \\
\boldsymbol{Fil}(X) \arrow[r, shift right, "\overrightarrow{f}"'] \arrow[r, phantom, "\perp"] & \boldsymbol{Fil}(Y) \arrow[l, shift right, "\overleftarrow{f}"']
\end{tikzcd}$$

[Holgate and Slapal, 2011]

Holgate, D. and Slapal, J. (2011).
Categorical neighborhood operators.
Top. Appl., 158:2356–2365.

Morphisms

Theorem 3.3 ([Holgate and Slapal, 2011, §3])

given the order preserving maps:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boldsymbol{Sub_M}(X)^{\mathrm{op}} \arrow[r, shift right, "\exists_f"'] \arrow[r, phantom, "\perp"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\geq"] & \boldsymbol{Sub_M}(Y)^{\mathrm{op}} \arrow[l, shift right, "f^{-1}"'] \arrow[d, "\phi"] \\
\boldsymbol{Fil}(X) \arrow[r, shift right, "\overrightarrow{f}"'] \arrow[r, phantom, "\perp"] & \boldsymbol{Fil}(Y) \arrow[l, shift right, "\overleftarrow{f}"']
\end{tikzcd}
$$

[Holgate and Slapal, 2011]

Holgate, D. and Slapal, J. (2011).
Categorical neighborhood operators.
Top. Appl., 158:2356–2365.

Internal Neighbourhood Structures

Definition 4.1 (Internal Weak Neighbourhood Spaces)
$\boldsymbol{wNbd}[\mathbb{A}]$ is the category whose objects are (X, μ), where μ is a weak neighbourhood on X and morphisms are $(X, \mu) \xrightarrow{f} (X, \phi)$ where f is a pre-neighbourhood morphism.

Internal Neighbourhood Structures

Definition 4.1 (Internal Pretopological Spaces)

$\boldsymbol{pre}[\mathbb{A}]$ is the category whose objects are (X, μ), where μ is a pre-neighbourhood on X and morphisms are $(X, \mu) \xrightarrow{f} (X, \phi)$ where f is a pre-neighbourhood morphism.

Internal Neighbourhood Structures

Definition 4.1 (Internal Pseudotopological Spaces)

$\boldsymbol{pse}[\mathbb{A}]$ is the category whose:

Internal Neighbourhood Structures

Definition 4.1 (Internal Pseudotopological Spaces)

(a) objects are $(X, \mathit{\Gamma})$,

Internal Neighbourhood Structures

Definition 4.1 (Internal Pseudotopological Spaces)

(a) objects are (X, Γ), where $\boldsymbol{Sub}_M(X) \xrightarrow{\Gamma} \boldsymbol{Fil}(\boldsymbol{Fil}(X))$ is an order preserving map such that for each $m \in \boldsymbol{Sub}_M(X)$:

Internal Neighbourhood Structures

Definition 4.1 (Internal Pseudotopological Spaces)

(a) objects are (X, Γ), where $\boldsymbol{Sub}_M(X) \xrightarrow{\Gamma} \boldsymbol{Fil}(\boldsymbol{Fil}(X))$ is an order preserving map such that for each $m \in \boldsymbol{Sub}_M(X)$:

$$\uparrow m \in \Gamma(m) \tag{7}$$

and

Internal Neighbourhood Structures

Definition 4.1 (Internal Pseudotopological Spaces)

(a) objects are (X, Γ), where $\boldsymbol{Sub_M}(X) \xrightarrow{\Gamma} \boldsymbol{Fil}(\boldsymbol{Fil}(X))$ is an order preserving map such that for each $m \in \boldsymbol{Sub_M}(X)$:

$$\uparrow m \in \Gamma(m) \tag{7}$$

and

$$(\forall A \in \boldsymbol{Fil}(X))\Big[(\forall U \in \boldsymbol{Max}[X]) \\ (A \subseteq U \Rightarrow U \in \Gamma(m)) \\ \Rightarrow A \in \Gamma(m)\Big] \tag{8}$$

Internal Neighbourhood Structures

Definition 4.1 (Internal Pseudotopological Spaces)

(b) morphisms are $(X, \Gamma) \xrightarrow{f} (Y, \Phi)$ where:

$$A \in \Gamma(m) \Rightarrow \overrightarrow{f} A \in \Phi(\exists_f m).$$

Internal Neighbourhood Structures

Definition 4.1 (Internal Neighbourhood Spaces)

$\boldsymbol{Nbd}[\mathbb{A}]$ is the category whose objects are (X, μ), where μ is a neighbourhood on X and morphisms are $(X, \mu) \xrightarrow{f} (Y, \phi)$, where f is a pre-neighbourhood morphism such that f^{-1} preserve arbitrary joins.

Internal Neighbourhood Structures

Theorem 4.1 (Equivalents of Pre-image Preserving Joins)

THE FOLLOWING ARE EQUIVALENT *for any morphism* $X \xrightarrow{f} Y$*:*

Internal Neighbourhood Structures

Theorem 4.1 (Equivalents of Pre-image Preserving Joins)

THE FOLLOWING ARE EQUIVALENT *for any morphism* $X \xrightarrow{f} Y$*:*

(a)
$$\begin{tikzcd} \boldsymbol{Sub_M}(Y) \arrow[rr, bend left, "f^{-1}"] \arrow[rr, phantom, "\perp"] & & \boldsymbol{Sub_M}(X) \arrow[ll, bend left, "\forall_f"] \end{tikzcd}$$

Internal Neighbourhood Structures

Theorem 4.1 (Equivalents of Pre-image Preserving Joins)

THE FOLLOWING ARE EQUIVALENT *for any morphism* $X \xrightarrow{f} Y$*:*

(a)

$$\begin{tikzcd} \boldsymbol{Sub_M}(Y) \arrow[rr, bend left, "f^{-1}"] \arrow[rr, phantom, "\perp"] & & \boldsymbol{Sub_M}(X) \arrow[ll, bend left, "\forall_f"] \end{tikzcd}$$

(b) f^{-1} *preserve joins.*

Internal Neighbourhood Structures

Theorem 4.1 (Equivalents of Pre-image Preserving Joins)

THE FOLLOWING ARE EQUIVALENT *for any morphism* $X \xrightarrow{f} Y$*:*

(a)
$$\begin{tikzcd} \boldsymbol{Sub_M}(Y) \arrow[rr, bend left, "f^{-1}"] \arrow[rr, phantom, "\perp"] & & \boldsymbol{Sub_M}(X) \arrow[ll, bend left, "\forall_f"] \end{tikzcd}$$

(b) f^{-1} *preserve joins.*

(c) $\overleftarrow{f}$ *preserve meets.*

Internal Neighbourhood Structures

Theorem 4.1 (Equivalents of Pre-image Preserving Joins)

THE FOLLOWING ARE EQUIVALENT *for any morphism* $X \xrightarrow{f} Y$*:*

(a)
$$\begin{tikzcd} \boldsymbol{Sub_M}(Y) \arrow[rr, bend left, "f^{-1}"] \arrow[rr, phantom, "\perp"] & & \boldsymbol{Sub_M}(X) \arrow[ll, bend left, "\forall_f"] \end{tikzcd}$$

(b) f^{-1} *preserve joins.*

(c) $\overleftarrow{f}$ *preserve meets.*

(d)
$$\begin{tikzcd} \boldsymbol{Fil}(X) \arrow[rr, bend left, "\overset{\sqcup}{f}"] \arrow[rr, phantom, "\perp"] & & \boldsymbol{Fil}(Y) \arrow[ll, bend left, "\overleftarrow{f}"] \end{tikzcd}$$

Internal Neighbourhood Structures

$$\boldsymbol{Sub_M}(Y) \xrightarrow{\quad f^{-1} \quad} \boldsymbol{Sub_M}(X)$$

$$\boldsymbol{Fil}(Y) \xrightarrow{\quad \overleftarrow{f} \quad} \boldsymbol{Fil}(X)$$

Internal Neighbourhood Structures

$$\begin{tikzcd} \boldsymbol{Sub_M}(Y) \arrow[rr, "f^{-1}"] \arrow[rr, phantom, shift left=3, "\perp"] & & \boldsymbol{Sub_M}(X) \arrow[ll, bend right, "\exists_f"'] \end{tikzcd}$$

$$\begin{tikzcd} \boldsymbol{Fil}(Y) \arrow[rr, "\overleftarrow{f}"] \arrow[rr, phantom, shift right=3, "\perp"] & & \boldsymbol{Fil}(X) \arrow[ll, bend left, "\overrightarrow{f}"] \end{tikzcd}$$

Internal Neighbourhood Structures

$$
\begin{tikzcd}
\boldsymbol{Sub_M}(Y) \arrow[rr, "f^{-1}" description] \arrow[rr, phantom, shift left=3, "\perp"] \arrow[rr, phantom, shift right=3, "\perp"] & & \boldsymbol{Sub_M}(X) \arrow[ll, bend right=20, "\exists_f"'] \arrow[ll, bend left=20, "\forall_f"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boldsymbol{Fil}(Y) \arrow[rr, "\overleftarrow{f}" description] \arrow[rr, phantom, shift left=3, "\perp"] \arrow[rr, phantom, shift right=3, "\perp"] & & \boldsymbol{Fil}(X) \arrow[ll, bend right=30, "\overset{\sqcup}{f}"'] \arrow[ll, bend left=30, "\overrightarrow{f}"]
\end{tikzcd}
$$

Internal Neighbourhood Structures

$$\forall_f x = \bigvee\{y \in \boldsymbol{Sub_M}(Y) : f^{-1}y \leq x\}$$

$$\begin{tikzcd}[column sep=huge]
\boldsymbol{Sub_M}(Y) \arrow[r, "f^{-1}" description] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \boldsymbol{Sub_M}(X) \arrow[l, bend right, "\exists_f"'] \arrow[l, bend left, "\forall_f"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
\boldsymbol{Fil}(Y) \arrow[r, "\overleftarrow{f}" description] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \boldsymbol{Fil}(X) \arrow[l, bend right, "\overset{\sqcup}{f}"'] \arrow[l, bend left, "\overrightarrow{f}"]
\end{tikzcd}$$

$$\overset{\sqcup}{f} A = \{y \in \boldsymbol{Sub_M}(Y) : (\exists a \in A)(\forall_f a \leq y)\}$$

Internal Neighbourhood Structures

Definition 4.1 (Internal Topological Spaces)

An *internal topological space* is an internal neighbourhood space (X, μ) in which $\mathfrak{O}_\mu$ is a frame in the partial order of $\boldsymbol{Sub_M}(X)$.

Internal Neighbourhood Structures

Definition 4.1 (Internal Topological Spaces)

$\boldsymbol{Top}[\mathbb{A}]$ is the full subcategory of $\boldsymbol{Nbd}[\mathbb{A}]$ consisting of internal topological spaces.

Internal Neighbourhood Structures

Definition 4.1 (The Non-full Subcategory of Preimage Preserve Joins)

$\mathbb{A}_{\boldsymbol{ppj}}$ is the (non-full) subcategory of $\mathbb{A}$ whose objects are same as of $\mathbb{A}$ and morphisms are those morphisms f from $\mathbb{A}$ for which f^{-1} preserve joins.

Reflectivity & Topologicity Chart

$$
\begin{tikzcd}
\boldsymbol{pse}[\mathbb{A}] & \boldsymbol{pre}[\mathbb{A}] & \boldsymbol{wNbd}[\mathbb{A}] & \boldsymbol{Nbd}[\mathbb{A}] \arrow[l, "\text{non-full subcategory}"'] \\
 & & & \boldsymbol{Top}[\mathbb{A}] \\
 & & \mathbb{A} & \mathbb{A}_{\boldsymbol{ppj}} \arrow[l, "\text{non-full subcategory}"]
\end{tikzcd}
$$

Reflectivity & Topologicity Chart

$$\begin{tikzcd}
\boldsymbol{pse}[\mathbb{A}] & \boldsymbol{pre}[\mathbb{A}] \arrow[ddr, bend right, "\text{topological}"'] & \boldsymbol{wNbd}[\mathbb{A}] \arrow[l, "\text{reflective}"'] \arrow[dd, "\text{topological}"] & \boldsymbol{Nbd}[\mathbb{A}] \arrow[l, "\text{non-full subcategory}"'] \arrow[dd, bend left, "\text{topological}"] \\
 & & & \boldsymbol{Top}[\mathbb{A}] \\
 & & \mathbb{A} & \mathbb{A}_{\boldsymbol{ppj}} \arrow[l, "\text{non-full subcategory}"]
\end{tikzcd}$$

Reflectivity & Topologicity Chart

$x \wedge \neg x = 0$

$$
\begin{tikzcd}
\boldsymbol{pse}[\mathbb{A}] & \boldsymbol{pre}[\mathbb{A}] \arrow[l, "\text{reflective}"'] \arrow[ddr, bend right, "\text{topological}"'] & \boldsymbol{wNbd}[\mathbb{A}] \arrow[l, "\text{reflective}"'] \arrow[dd, "\text{topological}"] & \boldsymbol{Nbd}[\mathbb{A}] \arrow[l, "\text{non-full subcategory}"'] \arrow[dd, bend left, "\text{topological}"] \\
 & & & \boldsymbol{Top}[\mathbb{A}] \\
 & & \mathbb{A} & \mathbb{A}_{\boldsymbol{ppj}} \arrow[l, "\text{non-full subcategory}"]
\end{tikzcd}
$$

Reflectivity & Topologicity Chart

Theorem 5.1 (Reflective Inclusion of $\boldsymbol{pre}[\mathbb{A}]$ in $\boldsymbol{pse}[\mathbb{A}]$)

If: each $\boldsymbol{Sub}_M(X)$ is further assumed to be pseudocomplemented,

Reflectivity & Topologicity Chart

Theorem 5.1 (Reflective Inclusion of $\boldsymbol{pre}[\mathbb{A}]$ in $\boldsymbol{pse}[\mathbb{A}]$)

If: *each $\boldsymbol{Sub}_M(X)$ is further assumed to be pseudocomplemented,*
then: $\boldsymbol{pre}[\mathbb{A}]$ is a reflective full subcategory of $\boldsymbol{pse}[\mathbb{A}]$.

Reflectivity & Topologicity Chart

$x \wedge \neg x = 0$

$$\begin{tikzcd}
\boldsymbol{pse}[\mathbb{A}] \arrow[ddrr, bend right, "\text{topological}"'] & \boldsymbol{pre}[\mathbb{A}] \arrow[l, "\text{reflective}"'] \arrow[ddr, bend right, "\text{topological}"'] & \boldsymbol{wNbd}[\mathbb{A}] \arrow[l, "\text{reflective}"'] \arrow[dd, "\text{topological}"] & \boldsymbol{Nbd}[\mathbb{A}] \arrow[l, "\text{non-full subcategory}"'] \arrow[dd, bend left, "\text{topological}"] \\
 & & & \boldsymbol{Top}[\mathbb{A}] \\
 & & \mathbb{A} & \mathbb{A}_{\boldsymbol{ppj}} \arrow[l, "\text{non-full subcategory}"]
\end{tikzcd}$$

$x \wedge f^{-1}y = 0 \Rightarrow$
$\exists_f x \wedge y = 0$

Reflectivity & Topologicity Chart

Theorem 5.1 (Topologicity of $\boldsymbol{pse}[\mathbb{A}]$)

Reflectivity & Topologicity Chart

Theorem 5.1 (Topologicity of $\boldsymbol{pse}[\mathbb{A}]$)

If for every morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$, each $x \in \boldsymbol{Sub}_M(X)$ and $y \in \boldsymbol{Sub}_M(Y)$:

$$x \wedge f^{-1} y = 0 \Rightarrow \exists_f x \wedge y = 0$$

then, $\boldsymbol{pse}[\mathbb{A}]$ is topological over $\mathbb{A}$.

Reflectivity & Topologicity Chart

Theorem 5.1 (Topologicity of $\boldsymbol{pse}[\mathbb{A}]$)

If for every morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$, each $x \in \boldsymbol{Sub_M}(X)$ and $y \in \boldsymbol{Sub_M}(Y)$:

$$x \wedge f^{-1} y = 0 \Rightarrow \exists_f x \wedge y = 0$$

then, $\boldsymbol{pse}[\mathbb{A}]$ is topological over $\mathbb{A}$.

Condition $x \wedge f^{-1} y = 0 \Rightarrow \exists_f x \wedge y = 0$

Reflectivity & Topologicity Chart

Theorem 5.1 (Topologicity of $\boldsymbol{pse}[\mathbb{A}]$)

If for every morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$, each $x \in \boldsymbol{Sub_M}(X)$ and $y \in \boldsymbol{Sub_M}(Y)$:

$$x \wedge f^{-1} y = 0 \Rightarrow \exists_f x \wedge y = 0$$

then, $\boldsymbol{pse}[\mathbb{A}]$ is topological over $\mathbb{A}$.

Condition $x \wedge f^{-1} y = 0 \Rightarrow \exists_f x \wedge y = 0$ yields:

Reflectivity & Topologicity Chart

Theorem 5.1 (Topologicity of $\boldsymbol{pse}[\mathbb{A}]$)
If for every morphism $X \xrightarrow{f} Y$ *of* $\mathbb{A}$*, each* $x \in \boldsymbol{Sub_M}(X)$ *and* $y \in \boldsymbol{Sub_M}(Y)$*:*

$$x \wedge f^{-1} y = 0 \Rightarrow \exists_f x \wedge y = 0$$

then, $\boldsymbol{pse}[\mathbb{A}]$ *is topological over* $\mathbb{A}$*.*

Condition $x \wedge f^{-1} y = 0 \Rightarrow \exists_f x \wedge y = 0$ yields:

(a) For every proper filter $A \in \boldsymbol{Fil}(X)$, $B \in \boldsymbol{Fil}(Y)$:

$$\overrightarrow{f} A \subseteq B \Rightarrow (\exists C \in \boldsymbol{Fil}(X))\big(A \subseteq C \text{ and } B \subseteq \overrightarrow{f} C\big).$$

Reflectivity & Topologicity Chart

Theorem 5.1 (Topologicity of $\boldsymbol{pse}[\mathbb{A}]$)
If for every morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$, each $x \in \boldsymbol{Sub_M}(X)$ and $y \in \boldsymbol{Sub_M}(Y)$:

$$x \wedge f^{-1} y = 0 \Rightarrow \exists_f x \wedge y = 0$$

then, $\boldsymbol{pse}[\mathbb{A}]$ is topological over $\mathbb{A}$.

Condition $x \wedge f^{-1} y = 0 \Rightarrow \exists_f x \wedge y = 0$ yields:

(a) For every proper filter $A \in \boldsymbol{Fil}(X)$, $B \in \boldsymbol{Fil}(Y)$:

$$\vec{f} A \subseteq B \Rightarrow (\exists C \in \boldsymbol{Fil}(X))\big(A \subseteq C \text{ and } B \subseteq \vec{f} C\big).$$

(b) For every maximal filter $U \in \boldsymbol{Max}[X]$, $\vec{f} U$ is a maximal filter on Y.

Reflectivity & Topologicity Chart

Theorem 5.1 (Topologicity of $\boldsymbol{pse}[\mathbb{A}]$)
If for every morphism $X \xrightarrow{f} Y$ *of* $\mathbb{A}$*, each* $x \in \boldsymbol{Sub_M}(X)$ *and* $y \in \boldsymbol{Sub_M}(Y)$*:*

$$x \wedge f^{-1} y = 0 \Rightarrow \exists_f x \wedge y = 0$$

then, $\boldsymbol{pse}[\mathbb{A}]$ *is topological over* $\mathbb{A}$*.*

Condition $x \wedge f^{-1} y = 0 \Rightarrow \exists_f x \wedge y = 0$ yields:

(a) For every proper filter $A \in \boldsymbol{Fil}(X)$, $B \in \boldsymbol{Fil}(Y)$:

$$\overrightarrow{f} A \subseteq B \Rightarrow (\exists C \in \boldsymbol{Fil}(X))\big(A \subseteq C \text{ and } B \subseteq \overrightarrow{f} C\big).$$

(b) For every maximal filter $U \in \boldsymbol{Max}[X]$, $\overrightarrow{f} U$ is a maximal filter on Y.

(c) $(X, \Gamma) \xrightarrow{f} (Y, \Phi)$ is a pseudotopological morphism, if and only if:

$$U \in \boldsymbol{Max}[X] \cap \Gamma(m) \Rightarrow \overrightarrow{f} U \in \boldsymbol{Max}[Y] \cap \Phi(\exists_f m).$$

Reflectivity & Topologicity Chart

$x \wedge \neg x = 0$

$$
\begin{tikzcd}
\boldsymbol{pse}[\mathbb{A}] \arrow[ddrr, bend right, "\text{topological}"'] & \boldsymbol{pre}[\mathbb{A}] \arrow[l, "\text{reflective}"'] \arrow[ddr, bend right, "\text{topological}"'] & \boldsymbol{wNbd}[\mathbb{A}] \arrow[l, "\text{reflective}"'] \arrow[dd, "\text{topological}"] & \boldsymbol{Nbd}[\mathbb{A}] \arrow[l, "\substack{\text{non-full}\\ \text{subcategory}}"'] \arrow[dd, bend left, "\text{topological}"] \\
 & & & \boldsymbol{Top}[\mathbb{A}] \arrow[u, "\text{reflective}"] \arrow[d, "\text{topological}"] \\
 & & \mathbb{A} & \mathbb{A}_{\boldsymbol{ppj}} \arrow[l, "\substack{\text{non-full}\\ \text{subcategory}}"]
\end{tikzcd}
$$

existence of largest topology

$x \wedge f^{-1}y = 0 \Rightarrow \exists_f x \wedge y = 0$

Reflectivity & Topologicity Chart

Theorem 5.1 (Reflective Inclusion of $\boldsymbol{Top}[\mathbb{A}]$ and topologicity)

THE FOLLOWING ARE EQUIVALENT :

Reflectivity & Topologicity Chart

Theorem 5.1 (Reflective Inclusion of $\boldsymbol{Top}[\mathbb{A}]$ and topologicity)

THE FOLLOWING ARE EQUIVALENT :

(a) For every object X, there exists a largest internal topology on X.

Reflectivity & Topologicity Chart

Theorem 5.1 (Reflective Inclusion of $\boldsymbol{Top}[\mathbb{A}]$ and topologicity)

THE FOLLOWING ARE EQUIVALENT :

(a) For every object X, there exists a largest internal topology on X.

(b) $\boldsymbol{Top}[\mathbb{A}]$ is a full reflective subcategory of $\boldsymbol{Nbd}[\mathbb{A}]$.

Reflectivity & Topologicity Chart

Theorem 5.1 (Reflective Inclusion of $\boldsymbol{Top}[\mathbb{A}]$ and topologicity)

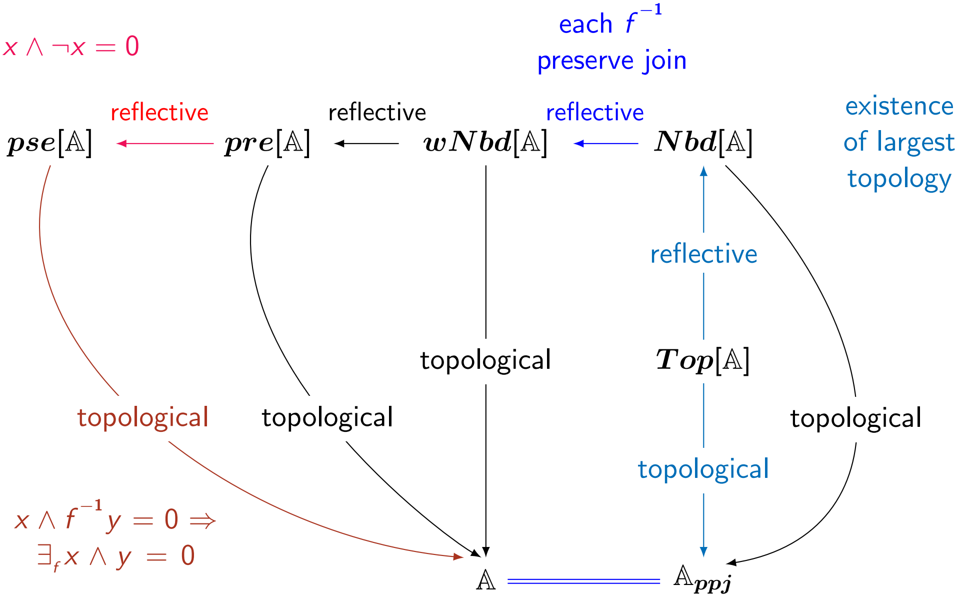
THE FOLLOWING ARE EQUIVALENT :

(a) *For every object X, there exists a largest internal topology on X.*

(b) $\boldsymbol{Top}[\mathbb{A}]$ *is a full reflective subcategory of* $\boldsymbol{Nbd}[\mathbb{A}]$.

(c) $\boldsymbol{Top}[\mathbb{A}]$ *is topological over* $\mathbb{A}_{\boldsymbol{ppj}}$.

Reflectivity & Topologicity Chart

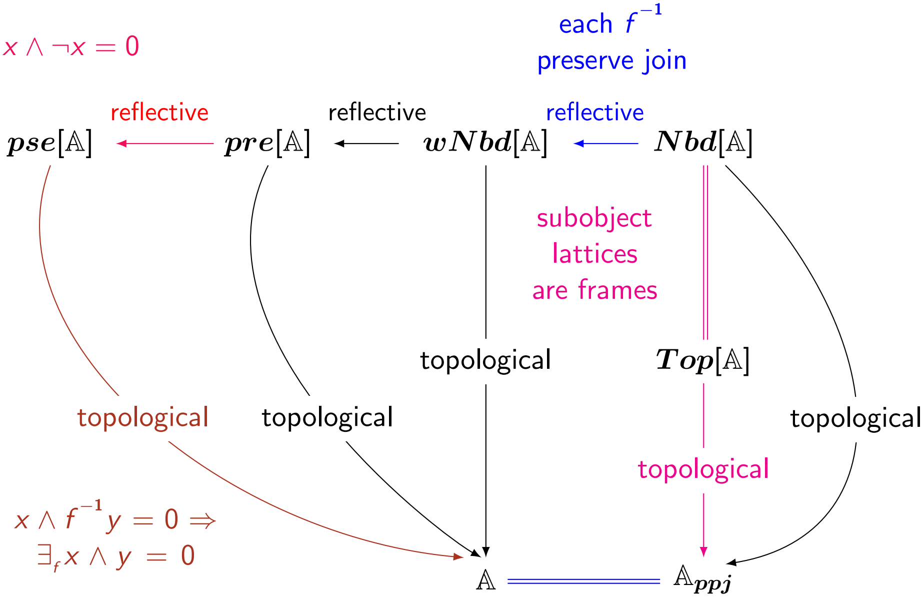


Reflectivity & Topologicity Chart

Theorem 5.1 (Reflectivity of $\boldsymbol{Nbd}[\mathbb{A}]$ in $\boldsymbol{wNbd}[\mathbb{A}]$)

If for every morphism $X \xrightarrow{f} Y$ of $\mathbb{A}$ f^{-1} preserve joins then $\boldsymbol{Nbd}[\mathbb{A}]$ is a full reflective subcategory of $\boldsymbol{wNbd}[\mathbb{A}]$.

Reflectivity & Topologicity Chart



Regular Epimorphisms of Internal Pretopological Spaces

Theorem 6.1

If the forgetful functor $\boldsymbol{pre}[\mathbb{A}] \xrightarrow{V} \mathbb{A}$ *create kernel pairs and preserve coequalisers then a morphism* $(X, \gamma) \xrightarrow{f} (Y, \phi)$ *of* $\boldsymbol{pre}[\mathbb{A}]$ *is a regular epimorphism, if and only if,* $X \xrightarrow{f} Y$ *is a regular epimorphism of* $\mathbb{A}$ *and:*

$$\phi(y) = \left\{u \in \boldsymbol{Sub}_M(Y) : y \leq u \text{ and } f^{-1}u \in \gamma(f^{-1}y)\right\}.$$

Regular Epimorphisms of Internal Pseudotopological Spaces

Theorem 6.2

If the forgetful functor $\boldsymbol{pre}[\mathbb{A}] \xrightarrow{W} \mathbb{A}$ *is topological then a morphism* $(X, \Gamma) \xrightarrow{f} (Y, \Phi)$ *of* $\boldsymbol{pre}[\mathbb{A}]$ *is a regular epimorphism, if and only if,* $X \xrightarrow{f} Y$ *is a regular epimorphism of* $\mathbb{A}$ *and:*

$$(\forall y \in \boldsymbol{Sub_M}(Y))(\forall V \in \boldsymbol{Max}[Y] \cap \Phi(y)) \\ (\exists x \in f^{-1} y)(\exists U \in \boldsymbol{Max}[X] \cap \Gamma(x)) \\ \left(V = \overrightarrow{f}\, U\right).$$

Way Forward. . .

(a) regular epimorphisms of internal neighbourhood spaces. . .

Way Forward. . .

(a) regular epimorphisms of internal neighbourhood spaces. . .

(b) how good are the extensions — topological hull? or quasi-topos hull? . . .

Way Forward...

(a) regular epimorphisms of internal neighbourhood spaces...
(b) how good are the extensions — topological hull? or quasi-topos hull? ...
(c) effective descent morphisms of the internal neighbourhood spaces

Way Forward...

(a) regular epimorphisms of internal neighbourhood spaces...

(b) how good are the extensions — topological hull? or quasi-topos hull? ...

(c) effective descent morphisms of the internal neighbourhood spaces

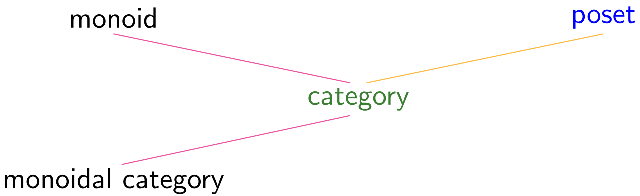
(d) possibilities of a general structure $(\mathbb{B}, \mathbb{B} \xrightarrow{F} \mathbb{A} \ldots)$ where F is a fibration and ...

Way Forward...

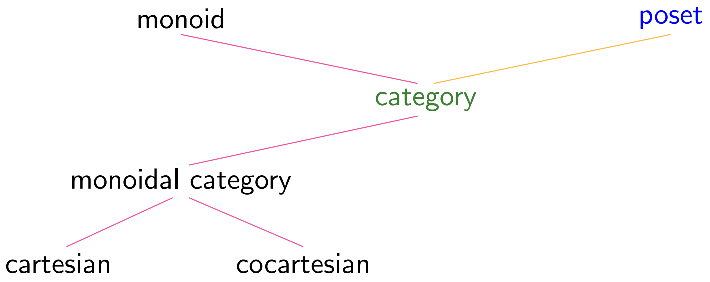
monoid poset

Way Forward. . .

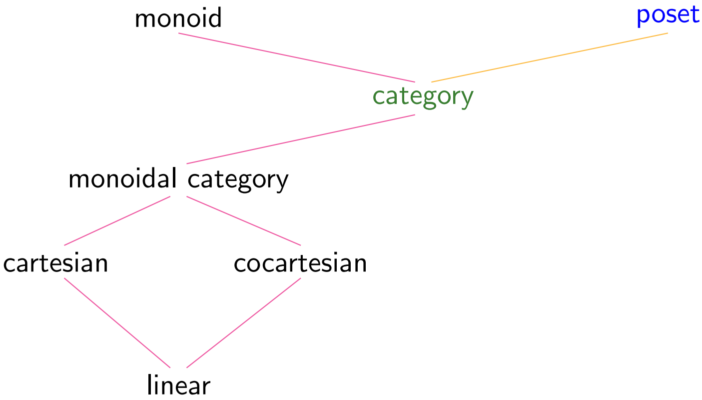
Way Forward. . .



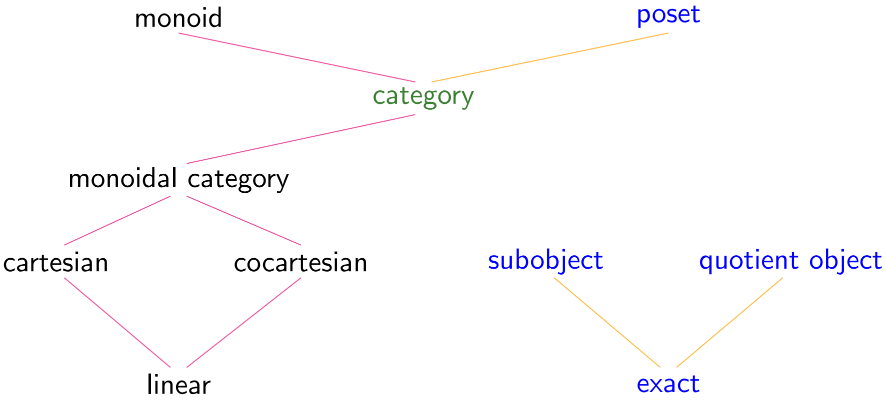
Way Forward. . .


monoid
poset
category
monoidal category
cartesian
cocartesian

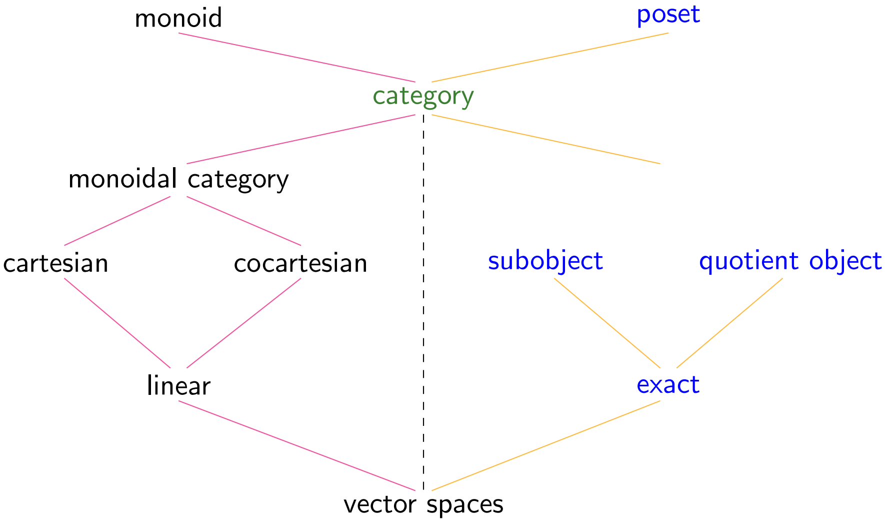
Way Forward. . .


monoid
poset
category
monoidal category
cartesian
cocartesian
linear

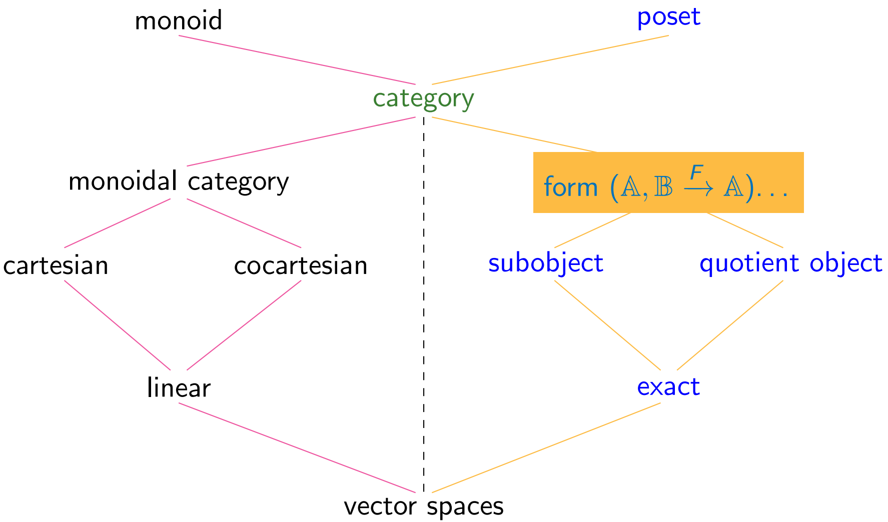
Way Forward. . .


monoid
poset
category
monoidal category
cartesian
cocartesian
subobject
quotient object
linear
exact

Way Forward. . .


monoid
poset
category
monoidal category
cartesian
cocartesian
subobject
quotient object
linear
exact
vector spaces

Way Forward...


monoid
poset
category
monoidal category
form $(\mathbb{A}, \mathbb{B} \xrightarrow{F} \mathbb{A})\ldots$
cartesian
cocartesian
subobject
quotient object
linear
exact
vector spaces

THANK YOU

Bentley, L., Herrlich, H., and Lowen, R. (1991).
Improving constructions in topology.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 3–20. Heldermann, Berlin.

Carboni, A., Janelidze, G., Kelly, G. M., and Pare, R. (1997).
On localization and stabilization for factorization systems.
Appl. Categ. Str., 5(1):1 – 58.

Carboni, A., Lack, S., and Walters, R. (1993).
Introduction to Extensive and Distributive Categories.
Jr. Pure Appl. Alg., 84:145 – 158.

Clementino, M. M., Giuli, E., and Tholen, W. (2004).
A Functional Approach to General Topology.
In *Categorical Foundations*, number 97 in Encyclopedia Math. Appl., pages 103–163. Cambridge University Press.

Herrlich, H., Lowen-Colebunders, E., and Schwarz, F. (1991).
Improving $\boldsymbol{Top}$: $\boldsymbol{PrTop}$ and $\boldsymbol{PsTop}$.
In Herrlich, H. and Porst, H., editors, *Category Theory at Work*, volume 18 of *Research Expositions in Mathematics*, pages 21–34. Heldermann, Berlin.

Holgate, D. and Slapal, J. (2011).

Categorical neighborhood operators.
Top. Appl., 158:2356–2365.

Iberkleid, W. and McGovern, W. W. (2009).
A Natural Equivalence for the Category of Coherent Frames.
Alg. Univ., 62:247–258.
available at: http://home.fau.edu/wmcgove1/web/Papers/WolfNatEq.pdf.

Kowalsky, H. (1965).
General Topology.
Academic Press, New York.