Hopf Formulae for Tor

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Main Point

Situation:

Birkhoff subcat

$$\text{R-Mod} \underset{\mathrm{Hom}(S,-)}{\overset{S\otimes_R -}{\rightleftarrows}} \text{S-Mod}$$

left modules

$I \rightarrowtail R \xrightarrow{\varphi} S$ ring hom induces R-module structure on S-modules

Given projective presentation of N

$$K \rightarrowtail P \twoheadrightarrow N \qquad H_2(N, S\otimes_R -) = \frac{IP \cap K}{IK}$$

$IM = \langle im \mid i \in I, m \in M \rangle$

Second group homology Hopf

$K \rightarrowtail P \twoheadrightarrow G$ projective presentation

$$H_2(G, \mathbb{Z}) = \frac{K \cap [P,P]}{[K,P]}$$

Commutators

Groups

$$\mathsf{Gp} \underset{\supseteq}{\overset{\mathrm{ab}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Ab}$$

$[P,P] \rightarrowtail P \twoheadrightarrow \mathrm{ab}\, P = \frac{P}{[P,P]}$

Semi-abelian

Semi-abelian $\mathcal{A} \underset{\supseteq}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{B}$ Birkhoff subcat

$[A]_{\mathcal{B}} \rightarrowtail A \xrightarrow{\eta_A} FA$

$$\begin{tikzcd} \mathrm{Ext}\,\mathcal{A} \arrow[r, shift left=1.5, "\text{centr.}"] \arrow[r, phantom, "\perp"] & \mathrm{CExt}\,\mathcal{A} \arrow[l, shift left=1.5, "2"] \end{tikzcd}$$

$$\begin{tikzcd} {[K,P]} \arrow[r, tail] \arrow[d, two heads] & P \arrow[r] \arrow[d, two heads, "p"] & P/[K,P] \arrow[d, two heads, "\mathrm{centr}(p)"] \\ 0 \arrow[r] & G \arrow[r, equal] & G \end{tikzcd}$$

central $\Leftrightarrow$
Kernel is in centre
of the group.

$$\begin{tikzcd} {[f]_{\mathrm{CExt}\,\mathcal{A}}} \arrow[r, tail] & f \arrow[r, two heads] & \mathrm{centr}(f) \end{tikzcd}$$

Given $\quad K \rightarrowtail P \xrightarrow{p} A \quad$ projective presentation

$$\rightsquigarrow \quad H_2(A, F) = \frac{[P]_{\mathcal{B}} \cap K}{\mathrm{Ker}\left([p]_{\mathrm{CExt}\,\mathcal{A}}\right)}$$

Everaert, Gran,
Van der Linden

Now specialise

F <u>protoadditive</u> = preserves split short exact sequences

$\Rightarrow$ central extensions are exactly $K \rightarrowtail A \xrightarrow{f} B$

where $K \in \mathcal{B}$

Everaert, Gran.

$\Rightarrow$ centralisation becomes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ A/[K]_{\mathcal{B}} \arrow[ur, "\mathrm{cent}(f)"'] & \end{tikzcd}$$

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & S \otimes_R M \\ m & \longmapsto & 1 \otimes m \end{array}$$

Kernel?

Apply $- \otimes_R M$ (also right exact!) to $0 \to I \rightarrowtail R \xrightarrow{\varphi} S \to 0$

$$\begin{array}{ccc} 1 \otimes rm & & \\ r \otimes m & \longmapsto & \varphi(r) \otimes m = 1 \otimes rm \\ \updownarrow & & \\ rm & & \end{array}$$

$$\Rightarrow \quad \begin{tikzcd} I \otimes_R M \arrow[r] \arrow[dr, two heads] & R \otimes_R M \arrow[r, "\eta_M"] \arrow[d, phantom, "\cong" rotate=90] & S \otimes_R M \arrow[r] & 0 \\ & \operatorname{Ker} \eta_M \arrow[u, tail] \quad M & & \end{tikzcd}$$

So $\operatorname{Ker} \eta_M = \langle im \mid i \in I,\ m \in M \rangle = IM$

Given $K \rightarrowtail P \twoheadrightarrow N$ projective presentation

$$\Rightarrow \quad H_2(N, S \otimes_R -) = \operatorname{Tor}_1^R(N, S) = \frac{IP \cap K}{IK}.$$

More special

$$\text{R-Mod} \underset{\supseteq}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{B} \qquad \text{R-Mod} \underset{\mathrm{Hom}(-,S)}{\overset{S\otimes_R -}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \text{S-Mod}$$

Subvariety

$$I \rightarrowtail R \xrightarrow{\varphi} S \quad \text{surjective rig hom}$$

→ any S-module becomes R-module

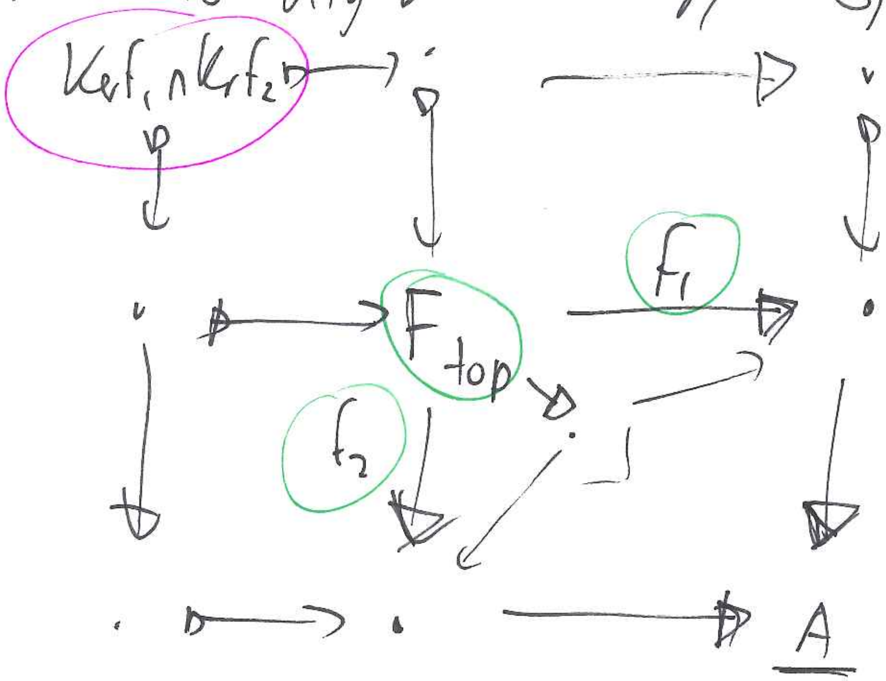
⇒ φ module hom SES in R-Mod

$S\otimes_R -$ is a left adjoint ⇒ right exact

$\overset{\text{abelian}}{\Longrightarrow}$ preserves <u>split</u> SES ⇒ protoadditive

⇒ centralisation becomes the nice form.

Extend to high homology objects



$f_1 \dots \quad f_n$

"initial ribs" of n-cube

$$H_{n+1}(A,F) = \mathrm{Tor}_n^R(A,S) = \frac{I F_{Top} \cap \left(\bigcap_{i=1}^{n} \operatorname{Ker} f_i\right)}{I\left(\bigcap_{i=1}^{n} \operatorname{Ker} f_i\right)}$$

Examples

(1) Group homology via $\quad G$ group $\quad M$ G-module

$R = \mathbb{Z}(G)$ group ring $\qquad S = \mathbb{Z}$

$\varphi : R \longrightarrow S$ augmentation map

$$I \rightarrowtail \mathbb{Z}(G) \longrightarrow \mathbb{Z} \qquad \Rightarrow H_n(G, M) = \mathrm{Tor}_n^R(M, S)$$

augmentation ideal

$$\sum n g \longmapsto \sum n$$

$$= H_{n+1}(M, S \otimes_R -)$$

$$= \frac{I F_{top} \cap (\bigcap \ker f_i)}{I (\bigcap \ker f_i)}$$

$M = \mathbb{Z}$ $\qquad I \longrightarrow \mathbb{Z}(G) \longrightarrow \mathbb{Z}$

$$H_1(G, \mathbb{Z}) = \frac{I\mathbb{Z}(G) \cap I}{II} = I/I^2 \cong \mathrm{ab}\, G$$

References

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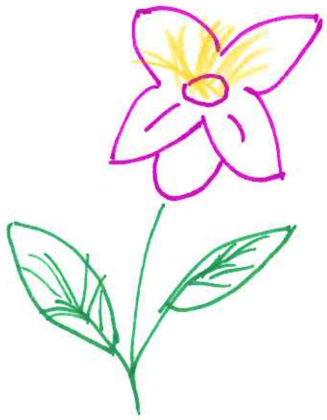

Thank you!