Central extensions in the variety of quandles

Marino Gran (Université catholique de Louvain)

joint work with Valérian Even and Andrea Montoli

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Quandles

Symmetric and abelian quandles

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- $(a \lhd b) \lhd^{-1} b = a, \quad (a \lhd^{-1} b) \lhd b = a$;
- $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$.

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If A and B are quandles, a quandle homomorphism $f\colon A \to B$ is a function such that

$$f(a \triangleleft a') = f(a) \triangleleft f(a'), \qquad f(a \triangleleft^{-1} a') = f(a) \triangleleft^{-1} f(a').$$

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We write Qnd for the category of quandles.

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We write Qnd^* for the category of trivial quandles.

Quandle associated with a group

Example

If $(G, \cdot, 1)$ is a group, one sets

$$g \lhd h = h^{-1} \cdot g \cdot h, \quad g \lhd^{-1} h = h \cdot g \cdot h^{-1} \qquad \forall g,\ h \in G.$$

This defines a quandle $Conj(G) = (G, \lhd, \lhd^{-1})$, called the *conjugation quandle* of G.

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Remark

A quandle can be seen as ... "what remains of a group when one only keeps the conjugation operation".

Any identity holding in $Conj(G)$ for all $G \in$ Grp also holds in Qnd.

Connected components of a quandle

For b in a quandle $(A, \lhd, \lhd^{-1})$, the right translation

$$\rho_b \colon A \to A$$

defined by

$$\rho_b(a) = a \lhd b, \quad \forall a \in A$$

is an automorphism.

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Let $\mathrm{Inn}(A)$ be the subgroup of $\mathrm{Aut}(A)$ generated by the right translations ρ_b :

$$\mathrm{Inn}(A) = \langle \{\rho_b \mid b \in A\} \rangle_{\mathrm{Aut}(A)}$$

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Two elements a and b of a quandle A are in the same connected component if there are $a_1, \ a_2, \ldots, a_n \in A$, $\lhd^{\alpha_i} \in \{\lhd, \lhd^{-1}\}$ such that

$$(\ldots((a \lhd^{\alpha_1} a_1) \lhd^{\alpha_2} a_2) \ldots) \lhd^{\alpha_n} a_n = b.$$

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A quandle A is algebraically connected if it has one connected component.

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A quandle A is algebraically connected if it has one connected component.

For any A, the set $\pi_0(A)$ of connected components is a trivial quandle.

The adjunction between Qnd and Qnd*

The functor $\pi_0 \colon$ Qnd $\to$ Qnd* is left adjoint to the inclusion functor $U \colon$ Qnd$^* \to$ Qnd

$$\begin{tikzcd} \mathrm{Qnd} \arrow[r, bend left, "\pi_0"] \arrow[r, phantom, "\perp"] & \mathrm{Qnd}^* \arrow[l, bend left, "U"] \end{tikzcd}$$

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Trivial quandles are determined by the additional identities

$$a \triangleleft b = a = a \triangleleft^{-1} b.$$

V. Even (2014) proved that the adjunction

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is admissible from the point of view of Categorical Galois Theory :

the reflection $\pi_0 \colon$ Qnd $\to$ Qnd* preserves all pullbacks of the form

$$\begin{tikzcd} A \times_{U(C)} U(B) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & U(B) \arrow[d, two heads, "U(g)"] \\ A \arrow[r, two heads, "f"'] & U(C) \end{tikzcd}$$

where $g \colon B \to C$ is a surjective homomorphism in Qnd*,
and $f \colon A \to U(C)$ is a surjective homomorphism in Qnd.

The categorical notion of covering arising from the adjunction

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gives back the notion of covering introduced by M. Eisermann (2014) :

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gives back the notion of covering introduced by M. Eisermann (2014) :

a surjective homomorphism $f\colon A \to B$ with the property that

$$f(x) = f(y) \text{ implies that } a \lhd x = a \lhd y, \text{ for any } a \in A.$$

Local permutability of congruences

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if R and S are congruences on a quandle A,

$$R \circ S \neq S \circ R,$$

in general.

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Definition

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Lemma (Even-Gran, 2014)

Let A be a quandle, R a congruence on A and $N \triangleleft \mathrm{Inn}(A)$. Then

$$\sim_N \circ R = R \circ \sim_N$$

Corollary

Let

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, two heads, "g"'] & B \arrow[d, two heads, "\overline{g}"] \\ C \arrow[r, two heads, "\overline{f}"'] & D \end{tikzcd}$$

be a pushout of surjective homomorphisms in Qnd,

$$\mathrm{Eq}(g) = \{(a, a') \mid g(a) = g(a')\} = \sim_N,$$

for a normal subgroup $N \triangleleft \mathrm{Inn}(A)$.

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be a pushout of surjective homomorphisms in Qnd,

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for a normal subgroup $N \lhd \mathrm{Inn}(A)$.

Then the induced morphism $(g, f)\colon A \to C \times_D B$ is surjective :

$$\begin{tikzcd} A \arrow[dr, dotted, two heads, "(g{,}f)"] \arrow[drr, bend left, two heads, "f"] \arrow[ddr, bend right, two heads, "g"'] & & \\ & C \times_D B \arrow[r, dotted, two heads] \arrow[d, dotted, two heads] & B \arrow[d, two heads, "\overline{g}"] \\ & C \arrow[r, two heads, "\overline{f}"'] & D \end{tikzcd}$$

Example

The kernel pair of the unit η_A

$$\sim_{\mathsf{Inn(A)}} = \mathsf{Eq}(\eta_A) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A \overset{\eta_A}{\dashrightarrow} U\pi_0(A)$$

is such a congruence.

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is such a congruence.

This fact plays a crucial role in the study of the Galois structure

$$\begin{tikzcd} \mathsf{Qnd} \arrow[rr, bend left, "\pi_0"] \arrow[rr, phantom, "\perp"] & & \mathsf{Qnd}^* \arrow[ll, bend left, "U"] \end{tikzcd}$$

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The subvariety of symmetric quandles

Qnd contains the subvariety SymQnd of symmetric quandles :

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A quandle Q is symmetric if it satisfies the additional identity

$$a \lhd b = b \lhd a, \qquad \forall a, b \in Q.$$

Lemma

The variety SymQnd of symmetric quandles is a Mal'tsev variety.

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Then :

$$p(a,a,b) = (a \lhd b) \lhd^{-1} a = (b \lhd a) \lhd^{-1} a = b,$$
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$\square$

Abelian quandles

A quandle A is abelian (or *medial*) if

$$(a \lhd b) \lhd (c \lhd d) = (a \lhd c) \lhd (b \lhd d).$$

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Remark

The notions of symmetric quandle and of abelian quandle are independent.

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for a counter-example, take the trivial quandle on $A = \{a, b\}$.

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Not all symmetric quandles are abelian :

the smallest counter-example has 81 elements (J.-P. Soublin).

The variety of abelian symmetric quandles

The Mal'tsev variety SymQnd contains the subvariety AbSymQnd of abelian symmetric quandles determined by the identity

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Lemma

AbSymQnd is a naturally Mal'tsev category (in the sense of P. Johnstone, 1989) :
any $A \in$ AbSymQnd has a natural Mal'tsev operation

$$p \colon A \times A \times A \to A.$$

There is a reflection

$$\begin{tikzcd}
\color{red}{\mathsf{Qnd}} \arrow[r, bend left, "sym"] \arrow[r, phantom, "\perp"] & \color{red}{\mathsf{SymQnd}} \arrow[l, bend left, "V"] \arrow[r, bend left, "ab"] \arrow[r, phantom, "\perp"] & \color{red}{\mathsf{AbSymQnd}} \arrow[l, bend left, "U"]
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similar to

$$\begin{tikzcd}[column sep=huge] \mathcal{V} \arrow[r, bend left, "I"] \arrow[r, phantom, "\perp"] & \mathcal{V}_{ab} \arrow[l, bend left, "U"] \end{tikzcd}$$

where $\mathcal{V}$ is a modular variety, $\mathcal{V}_{ab}$ its subvariety of abelian algebras.

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Lemma
The reflection

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is admissible for Categorical Galois Theory.

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is admissible for Categorical Galois Theory.

This means that the reflector $ab \circ sym\colon$ Qnd $\to$ AbSymQnd preserves the pullbacks of the form

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\begin{tikzcd}
A \times_{(V\circ U)(C)} (V \circ U)(B) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & (V \circ U)(B) \arrow[d, two heads, "(V\circ U)(g)"] \\
A \arrow[r, two heads, "f"'] & (V \circ U)(C)
\end{tikzcd}
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Indeed, each b of B determines a subquandle $\{b\}$ of B :

$$\begin{tikzcd}
f^{-1}(b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
\{b\} \arrow[r] & B
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We say that $f\colon A \to B$ has abelian symmetric fibers if each fiber $f^{-1}(b)$ is in AbSymQnd.

Fact :

If $f\colon A \to B$ has abelian symmetric fibers, then

$$Eq(f) \circ R = R \circ Eq(f)$$

for any congruence R on A.

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This also follows from a result by D. Bourn (2015) concerning some "partial Mal'tsev" properties of Qnd.

Definition

If R and S are equivalence relations on A, a double equivalence relation C on R and S

$$\begin{tikzcd} C \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A \end{tikzcd}$$

is called a centralizing relation if the square

$$\begin{tikzcd} C \arrow[r, "p_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "s_1"] \\ R \arrow[r, "r_2"'] & A \end{tikzcd}$$

is a pullback.

In a Mal'tsev variety the existence of a centralizing relation C on R and S

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is equivalent to the triviality of the Smith commutator :

$$[R, S] = \Delta_A,$$

In a Mal'tsev variety the existence of a centralizing relation C on R and S

$$\begin{array}{ccc} C & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & S \\ {\scriptstyle \pi_1}\downarrow\downarrow{\scriptstyle \pi_2} & & {\scriptstyle s_1}\downarrow\downarrow{\scriptstyle s_2} \\ R & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & A \end{array}$$

is equivalent to the triviality of the Smith commutator :

$$[R, S] = \Delta_A,$$

and to the existence of a partial Mal'tsev operation $p\colon R \times_A S \to A$:

$$p(x, y, y) = x, \qquad p(x, x, y) = y.$$

Definition

A surjective quandle homomorphism $f\colon A \to B$ is an algebraically central extension if there is a centralizing relation on $\mathrm{Eq}(f)$ and $A \times A$:

$$
\begin{tikzcd}
C \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & A \times A \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & \\
\mathrm{Eq}(f) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & A \arrow[r, dotted, "f"] & B
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Lemma

- For a quandle homomorphism $f\colon A \to B$ with abelian symmetric fibers such a centralizing relation is unique, when it exists.

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Lemma

- For a quandle homomorphism $f\colon A \to B$ with abelian symmetric fibers such a centralizing relation is unique, when it exists.
- When this is the case, one can prove that

$$\mathrm{Eq}(f) \cong A \times Q$$

where Q is an abelian symmetric quandle.

This type of products

$$\begin{tikzcd} A \times Q \arrow[r, two heads] \arrow[d, two heads] & Q \arrow[d, two heads] \\ A \arrow[r, two heads] & \{\star\} = 1 \end{tikzcd}$$

is preserved by the reflector $ab \circ sym \colon$ Qnd $\rightarrow$ AbSymQnd.

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is preserved by the reflector $ab \circ sym\colon$ Qnd $\to$ AbSymQnd.

This follows from the surjectivity of $Q \to 1$, and the fact that it lies in AbSymQnd.

A surjective homomorphism $f\colon A \to B$ in Qnd is a trivial covering if the commutative square induced by the units of the reflection

$$\begin{tikzcd}
A \arrow[r, two heads, "f"] \arrow[d, two heads, "\eta_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "\eta_B"] \\
(ab \circ sym)(A) \arrow[r, two heads, "(ab\circ sym)(f)"'] & (ab \circ sym)(B)
\end{tikzcd}$$

is a pullback.

A surjective homomorphism $f\colon A \to B$ in Qnd is a trivial covering if the commutative square induced by the units of the reflection

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, two heads, "\eta_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "\eta_B"] \\ (ab \circ sym)(A) \arrow[r, two heads, "(ab \circ sym)(f)"'] & (ab \circ sym)(B) \end{tikzcd}$$

is a pullback.

A surjective homomorphism $f\colon A \to B$ is a covering if there is a surjective homomorphism $p\colon E \to B$ such that $\pi_1 : E \times_B A \to E$ in

$$\begin{tikzcd} E \times_B A \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "f"] \\ E \arrow[r, two heads, "p"'] & B \end{tikzcd}$$

is a trivial covering.

Theorem (Even, Gran, Montoli, 2016)

Given a surjective homomorphism $f \colon A \to B$ in Qnd, the following conditions are equivalent :

1. f is a covering for the reflection

$$\begin{tikzcd} \mathsf{Qnd} \arrow[rr, bend left, "sym"] \arrow[rr, phantom, "\perp"] & & \mathsf{SymQnd} \arrow[ll, bend left, "V"] \arrow[rr, bend left, "ab"] \arrow[rr, phantom, "\perp"] & & \mathsf{AbSymQnd} \arrow[ll, bend left, "U"] \end{tikzcd} \quad (1)$$

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Remark

For a homomorphism the two conditions "being algebraically central" and "having abelian symmetric fibers" are independent.

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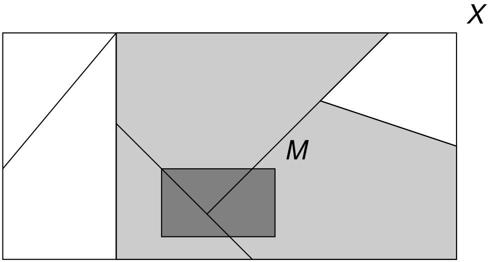
Other results concern closure operators in Qnd



and factorization systems associated with reflective subcategories.

Bibliography

- M. Eisermann, Quandle coverings and their Galois correspondence, *Fund. Mathematicae* (2014)
- E. Bunch, P. Lofgren, A. Rapp, and D. N. Yetter, On quotients of quandles, *J. Knot Theory Ramifications* (2010)
- V. Even and M. Gran, On factorization systems for surjective quandle homomorphisms, *J. Knot Theory Ramifications* (2014)
- V. Even and M. Gran, Closure operators in the category of quandles, *Topology and its Applications* (2016)
- V. Even, M. Gran and A. Montoli, A characterization of central extensions in the variety of quandles, *Theory and Applications of Categories* (2016)
- P.T. Johnstone, Affine categories and naturally Mal'cev categories, *J. Pure Appl. Algebra* (1989)
- D. Bourn, A structural aspect of the category of quandles, *J. Knot Theory Ramifications* (2015)