Duality theory, convergence, and enriched categories

Dirk Hofmann

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- Maria Manuel Clementino
- Renato Neves
- Pedro Nora
- Carla Reis
- Lurdes Sousa
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F. William Lawvere (1973). "Metric spaces, generalized logic, and closed categories". In: *Rendiconti del Seminario Matemàtico e Fisico di Milano* **43**.(1), pp. 135–166. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.

"While listening to a 1967 lecture of Richard Swan, which included a discussion of the relative codimension of pairs of subvarieties, I noticed the analogy between the triangle inequality and a categorical composition law.

Ordered sets and metric spaces

$$\begin{array}{ll} \top \implies (x \leq x), & (x \leq y \& y \leq z) \implies x \leq z \\ 0 \geqslant d(x,x), & d(x,y) + d(y,z) \geqslant d(x,z) \end{array}$$

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"While listening to a 1967 lecture of Richard Swan, which included a discussion of the relative codimension of pairs of subvarieties, I noticed the analogy between the triangle inequality and a categorical composition law. Later I saw that Hausdorff had mentioned the analogy between metric spaces and posets."

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Felix Hausdorff (1914). *Grundzüge der Mengenlehre*. Leipzig: Veit & Comp. VIII and 476 pages.

Thinking of an order relation on a set M as a function

$$f: M \times M \longrightarrow \{<, >, =\},$$

Hausdorff observes that

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Hausdorff observes that

"Nun steht einer Verallgemeinerung dieser Vorstellung nichts im Wege, und wir können uns denken, daß eine beliebige Funktion der Paare einer Menge definiert, d.h. jedem Paar (a, b) von Elementen einer Menge M ein bestimmtes Element n = f(a, b) einer zweiten Menge N zugeordnet sei. In noch weiterer Verallgemeinerung können wir eine Funktion der Elementtripel, Elementfolgen, Elementkomplexe, Teilmengen u. dgl. von M in Betracht ziehen."

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Now there stands nothing in the way of a generalisation of this idea, and we can think of an arbitrary function of pairs of points which associates to each pair (a, b) of elements of a set M a specific element n = f(a, b) of a second set N. Generalising further, we can consider a function of triples, sequences, complexes, subsets, etc.

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... but also writes:

"Eine ganz allgemein gehaltene Theorie dieser Art würde natürlich erhebliche Komplikationen bedingen und wenig positive Ausbeute liefern."

A very general theory of this kind would bring considerable complications and little benefit.

Hausdorff might think of a structure like $TX \times X \longrightarrow \mathcal{V}$...

... where $\mathbb{T} = (T, m, e)$ is a monad and $\mathcal{V} = (\mathcal{V}, \otimes, k)$ is a quantale!?

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Example

• A topology on X is a relation $a: UX \times X \longrightarrow \mathbf{2}$ so that

$$\top \implies (\dot{x} \xrightarrow{a} x) \quad \text{and} \quad (\mathfrak{X} \xrightarrow{Ua} \mathfrak{x} \& \mathfrak{x} \xrightarrow{a} x) \implies m_X(\mathfrak{X}) \xrightarrow{a} x \\ 1_X \leq a \cdot e_X \quad \text{and} \quad a \cdot Ua \leq a \cdot m_X \quad (Note : e_X \dashv e_X^\circ, \ m_X \dashv m_X^\circ).$$

Michael Barr (1970). "Relational algebras". In: *Reports of the Midwest Category Seminar, IV*. Lecture Notes in Mathematics, Vol. 137. Springer, Berlin, pp. 39–55.

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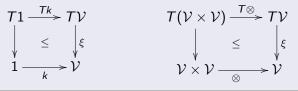
• A topology on X is a map $a: X \to FX$ so that $e_X < a$ and $a \circ a < a$.

Werner Gähler (1992). "Monadic topology – A new concept of generalized topology". In: *Recent developments of general topology and its applications.* Berlin: Akademie-Verlag, pp. 136–149. International conference in memory of Felix Hausdorff (1868 - 1942), held in Berlin, Germany, March 22-28, 1992.

Definition

A topological theory $\mathfrak{T} = (\mathbb{T}, \mathcal{V}, \xi)$ consists of a monad $\mathbb{T} = (T, m, e)$, a quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$ and a map $\xi \colon T\mathcal{V} \to \mathcal{V}$ so that

- *T* preserves weak pullbacks and each naturality square of *m* is a weak pullback;
- ξ: TV → V is the structure of a lax Eilenberg–Moore algebra on V and ξ is "compatible with suprema in V";
- the monoid operations are lax homomorphisms:

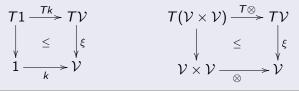


Dirk Hofmann (2007). "Topological theories and closed objects". In: *Advances in Mathematics* **215**.(2), pp. 789–824.

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Walter Tholen (2016). Lax Distributive Laws for Topology, I. Tech. rep. arXiv: 1603.06251 [math.CT].

... And Now for Something Completely Different!

Duality theory (discrete case)

 $\mathsf{CoAlg}(V) \simeq \mathbf{BAO}^{\mathrm{op}}$

Here $V: \operatorname{BooSp} \longrightarrow \operatorname{BooSp}$ is the Vietoris functor.

Bjarni Jónsson and Alfred Tarski (1951). "Boolean algebras with operators. I". In: *American Journal of Mathematics* **73**.(4), pp. 891–939.

Paul R. Halmos (1956). "Algebraic logic I. Monadic Boolean algebras". In: *Compositio Mathematica* **12**, pp. 217–249.

Clemens Kupke, Alexander Kurz, and Yde Venema (2004). "Stone coalgebras". In: *Theoretical Computer Science* **327**.(1-2), pp. 109–134.

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Marshall Harvey Stone (1936). "The theory of representations for Boolean algebras". In: Transactions of the American Mathematical Society 40.(1), pp. 37–111.

Duality theory (ordered case)

$$\begin{array}{c} \text{Priest} \xrightarrow{\text{hom}(-,2)} \to \text{DL}^{\operatorname{op}} \\ \downarrow & \downarrow \\ \\ \text{Priest}_{\mathbb{V}} \xrightarrow{} & \text{hom}(-,1) \\ \end{array} \xrightarrow{} \text{FinSup}^{\operatorname{op}} \end{array}$$

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Hilary A. Priestley (1970). "Representation of distributive lattices by means of ordered stone spaces". In: *Bulletin of the London Mathematical Society* **2**.(2), pp. 186–190.

Roberto Cignoli, S. Lafalce, and Alejandro Petrovich (1991). "Remarks on Priestley duality for distributive lattices". In: *Order* **8**.(3), pp. 299–315.

Alejandro Petrovich (1996). "Distributive lattices with an operator". In: *Studia Logica* **56**.(1-2), pp. 205–224. Special issue on Priestley duality. Marcello M. Bonsangue, Alexander Kurz, and Ingrid M. Rewitzky (2007). "Coalgebraic representations of distributive lattices with operators". In: *Topology and its Applications* **154**.(4), pp. 778–791.

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These results suggest:

We are looking for a(n at least) fully faithful functor

$$\mathsf{PosComp}_{\mathbb{V}} \xrightarrow{\text{``} \mathcal{C} = \mathsf{hom}(-,[0,1])''} (???)^{\mathrm{op}}$$

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- [0, 1]-categories with finite weighted colimits and finitely cocontinuous [0, 1]-functors;
- with monoid structure \otimes and neutral element 1, laxly preserved.

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$$\mathcal{V} = 2$$
 with $\otimes = \&$ and $k = \top$: 2-Cat \simeq Ord.
2. $\mathcal{V} = [0, \infty]_+$ with $\otimes = +$ and $k = 0$: $[0, \infty]_+$ -Cat \simeq Met.

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7. $\mathcal{V} = [0, 1]_{\odot}$ with $u \otimes v = u + v - 1$ and $k = 1$; $[0, 1]_{\odot} \simeq [0, 1]_{\oplus}$.

Examples

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8. $\mathcal{V} = \Delta = \{f : [0, \infty] \rightarrow [0, 1] \mid f(\alpha) = \bigvee_{\beta < \alpha} f(\beta)\}, \Delta$ -**Cat** \simeq **ProbMet**.

Karl Menger (1942). "Statistical metrics". In: Proceedings of the National Academy of Sciences of the United States of America 28, pp. 535–537.

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• tensor: $f \otimes g(\gamma) = \bigvee_{\alpha + \beta \leq \gamma} f(\alpha) * g(\beta),$

• neutral element: $f_{0,1}(0) = 0$, $f_{0,1}(\alpha) = 1$ for $\alpha > 0$.

The discrete case

The functor V:**CompHaus** \longrightarrow **CompHaus** is defined by

Leopold Vietoris (1922). "Bereiche zweiter Ordnung". In: *Monatshefte für Mathematik und Physik* **32**.(1), pp. 258–280.

The discrete case

The functor V:**CompHaus** \longrightarrow **CompHaus** is defined by

• $VX = \{A \subseteq X \mid A \text{ closed}\}$ with the "hit-and-miss topology" $\{A \mid A \cap B \neq \varnothing\}, \quad \{A \mid A \cap B^{\complement} = \varnothing\}$ (for all B open);

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Note: **PosComp** \simeq **StablyComp** \longrightarrow **Top**.

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The topological case

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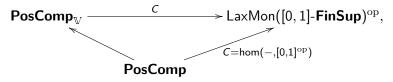
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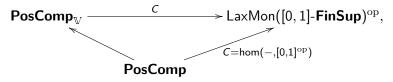
We consider:



The induced monad morphism j is precisely given by the family of maps $j_X \colon VX \longrightarrow [CX, [0, 1]], A \longmapsto \Phi_A,$ with $\Phi_A \colon CX \longrightarrow [0, 1], \psi \longmapsto \sup_{x \in A} \psi(x).$

Dirk Hofmann and Pedro Nora (2016). Enriched Stone-type dualities. Tech. rep. arXiv: 1605.00081 [math.CT].

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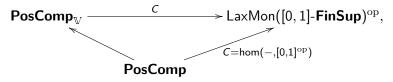
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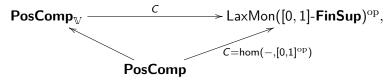
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^aWait till 09h43 $\pm \varepsilon$.

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For $\otimes = *$ multiplication resp. $\otimes = \odot$ the Łukasiewicz tensor,

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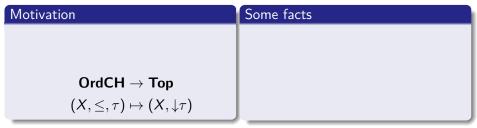
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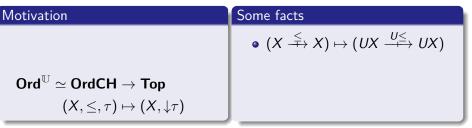
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- $\mathbb{T} = \mathbb{W}$ free monoid monad, $\xi : WV \to V, (v_1, \dots, v_n) \mapsto v_1 \otimes \dots \otimes v_n$: Here \mathcal{T} -**Cat** = V-**MultiCat**; for V = 2, \mathcal{T} -**Cat** = **MultiOrd**.



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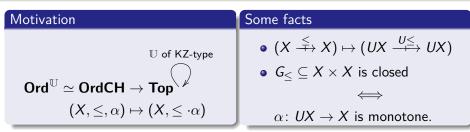
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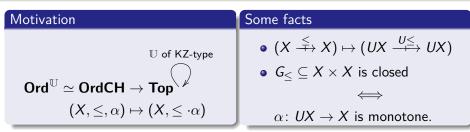
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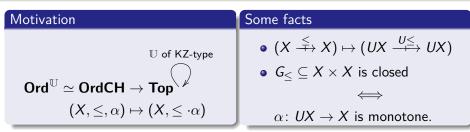


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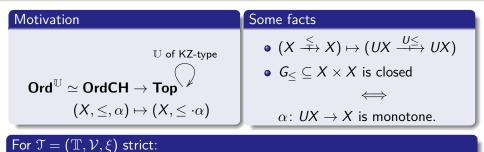
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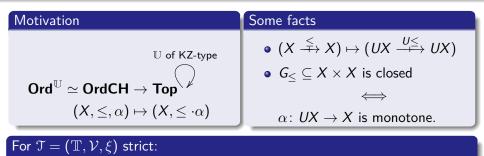
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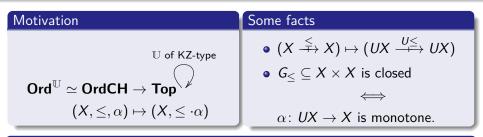


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MotivationSome facts \mathbb{U} of KZ-type• $(X \stackrel{\leq}{\longrightarrow} X) \mapsto (UX \stackrel{U \leq}{\longrightarrow} UX)$ $\mathbf{Ord}^{\mathbb{U}} \simeq \mathbf{OrdCH} \rightarrow \mathbf{Top}$ • $(X, \leq \cdot \alpha)$ $(X, \leq, \alpha) \mapsto (X, \leq \cdot \alpha)$ • $\alpha : UX \rightarrow X$ is monotone.

For $\mathfrak{T} = (\mathbb{T}, \mathcal{V}, \xi)$ strict:

$$(\mathcal{V} extsf{-Cat})^{\mathbb{T}} \underbrace{\overset{\mathcal{K}}{\overbrace{M}}}_{M} \mathcal{T} extsf{-Cat} \bigcirc \mathbb{T} \text{ of KZ-type}$$

- Representable T-category = pseudo-algebra for T on T-Cat.
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- Under some conditions: Representable \implies Cauchy-complete.

On cocomplete T-categories

For $\mathfrak{T} = (\mathbb{T}, \mathcal{V}, \xi)$ strict:

• $a: (TX)^{\mathrm{op}} \otimes X \longrightarrow \mathcal{V}$ is a \mathcal{V} -functor and there is an adjunction

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Examples

• For Top: $\mathbb{P} = \mathbb{F}$ is the filter monad, $\mathsf{Top}^{\mathbb{F}} \simeq \mathsf{Set}^{\mathbb{F}} \simeq \mathsf{ContLat}^a$.

^aDana Scott (1972). "Continuous lattices". In: *Toposes, algebraic geometry and logic (Conf., Dalhousie Univ., Halifax, N. S., 1971).* Vol. 274. Springer, Lect. Notes Math., pp. 97–136.

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^aJoachim Lambek, Michael Barr, John F. Kennison, and Robert Raphael (2012). "Injective hulls of partially ordered monoids". In: *Theory and Applications of Categories* **26**.(13), pp. 338–348.

Some notions and facts

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 with $e_X^\circ \leq a$.

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- T-graph = (X, a) with $e_X^\circ \leq a$.
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Theorem (for a dualisable T-category X)

 X^{op} is a \mathbb{T} -category $\iff X$ is core-compact^a $\iff X$ is representable.

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Theorem

This construction defines a monad $\mathbb{V} = (V, m, e)$ of co-KZ-type on \mathbb{T} -Cat and \mathbb{T} -Rep.

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For a study of (co)-Kock–Zöberlein monads in Topology:

Margarida Carvalho and Lurdes Sousa (2015). "On Kan-injectivity of Locales and Spaces". In: *Applied Categorical Structures*, pp. 1–22. Jiří Adámek, Lurdes Sousa, and Jiří Velebil (2015). "Kan injectivity in order-enriched categories". In: *Mathematical Structures in Computer Science* **25**.(1), pp. 6–45.

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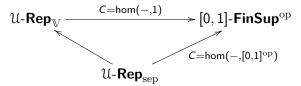
Example (in **Top**)

Let $\mathbb{D} = (D, m, e)$ be a co-Kock–Zöberlein monad on **Top** where each component e_X of e is an embedding. Then \mathbb{D} is a submonad of \mathbb{V} .

Now back to duality theory

Duality theory for representable U-distributors

We consider now $\mathcal{U} = (\mathbb{U}, [0, 1], \xi)$ with $\otimes = *$ or $\otimes = \odot$. Then

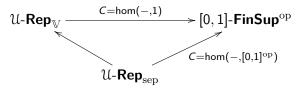


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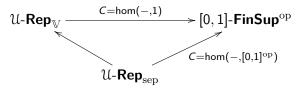
Theorem

[0,1] is an initial (=regular) cogenerator in **PosComp**.

Leopoldo Nachbin (1950). Topologia e Ordem. Univ. of Chicago Press.

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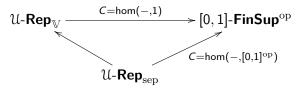
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Definition

A separated representable \mathcal{U} -category X is called $[0, 1]^{\mathrm{op}}$ -cogenerated if the cone $(\psi : X \to [0, 1]^{\mathrm{op}})_{\psi \in CX}$ is point-separating and initial.

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Proposition

X is $[0,1]^{\mathrm{op}}$ -cogenerated \implies VX is $[0,1]^{\mathrm{op}}$ -cogenerated.

Duality theory for representable \mathcal{U} -categories

Theorem

The functor

$$\textit{C} \colon \big(\mathfrak{U}\text{-}\mathsf{Rep}_{[0,1]^{\mathrm{op}}} \big)_{\mathbb{W}} \longrightarrow [0,1]\text{-}\mathsf{FinSup}^{\mathrm{op}}$$

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Let X be a partially ordered compact space and $\psi_0 \in CX$. Then $\psi_0 \otimes -: CX \longrightarrow CX$ is the largest morphism $\Phi: CX \longrightarrow CX$ in [0, 1]-**FinSup** satisfying

 $\Phi(1) \leq \psi_0$ and $\Phi(\psi) \leq \psi$ (for all $\psi \in CX$).

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A morphism $\varphi \colon X \Leftrightarrow Y$ in \mathcal{U} -**Rep**_V between partially ordered compact spaces is in **PosComp**_V if and only if $C\varphi$ preserves laxly the tensor.

Before

For
$$C:$$
 StablyComp_V \longrightarrow LaxMon([0, 1]-FinSup)^{op} and

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Recall: \mathfrak{T} -distributor $\varphi: X \Leftrightarrow Y = \varphi: TX \times Y \to \mathcal{V}$ so that

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Maria Manuel Clementino and Dirk Hofmann (2009). "Lawvere completeness in topology". In: *Applied Categorical Structures* **17**.(2), pp. 175–210.

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Examples

• In **Met**: Cauchy complete = Cauchy complete^a.

^aF. William Lawvere (1973). "Metric spaces, generalized logic, and closed categories". In: *Rendiconti del Seminario Matemàtico e Fisico di Milano* **43**.(1), pp. 135–166.

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X is Cauchy complete if every adjunction $\varphi \dashv \psi$ is induced by some $x \in X$.

Examples

- In Met: Cauchy complete = Cauchy complete.
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Recall:
$$\mathfrak{T}$$
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^aBernhard Banaschewski, Robert Lowen, and Cristophe Van Olmen (2006). "Sober approach spaces". In: *Topology and its Applications* **153**.(16), pp. 3059–3070.

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Proposition

Every representable U-category is Cauchy complete.

Proposition

An \mathcal{U} -distributor $\varphi: 1 \Leftrightarrow X$ is left adjoint \iff the [0,1]-functor $[\varphi, -]: \mathcal{U}$ -**Cat** $(X, [0,1]) \longrightarrow [0,1]$ preserves tensors and finite suprema.

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Remark

• Well-known: a \mathcal{V} -distributor $\varphi \colon 1 \to X$ is left adjoint $\iff [\varphi, -] \colon \mathcal{V}$ -**Cat** $(X, \mathcal{V}) \to \mathcal{V}$ preserves weighted colimits.

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Leopoldo Nachbin (1992). "Compact unions of closed subsets are closed and compact intersections of open subsets are open". In: *Portugaliæ Mathematica* **49**.(4), pp. 403–409.

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For Łukasiewicz $\otimes = \odot$

[0, 1] is a Girard quantale: for every $u \in [0, 1]$, $u = hom(hom(u, \perp), \perp)$ where $hom(u, \perp) = 1 - u =: u^{\perp}$. Furthermore, the diagram

$$egin{aligned} [0,1] extsf{-Dist}(X,1) & \stackrel{(-)^{\perp}}{\longrightarrow} [0,1] extsf{-Dist}(1,X)^{\mathrm{op}} \ & & \downarrow^{[arphi,-]^{\mathrm{op}}} \ & [0,1] & \stackrel{(-)^{\perp}}{\longrightarrow} [0,1]^{\mathrm{op}} \end{aligned}$$

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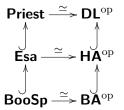
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commutes in [0,1]-**Cat** and $CX \hookrightarrow \mathcal{U}$ -**Cat** $(X, [0,1]^{\mathrm{op}})$ is \bigvee -dense. **Conclusion:** $\varphi: 1 \Leftrightarrow X$ is left adjoint $\iff \Phi$ preserves finite w. limits.

$$\mathsf{Priest} \xrightarrow{\simeq} \mathsf{DL}^{\mathrm{op}}$$

$$\mathsf{BooSp} \xrightarrow{\simeq} \mathsf{BA}^{\mathrm{op}}$$



Definition

A Priestley space (X, \leq, α) is called an Esakia space whenever $i: (X, \alpha) \rightarrow (X, \leq, \alpha)$ is downwards open.

Leo Esakia (1974). "Topological Kripke models". In: *Doklady Akademii* Nauk SSSR **214**, pp. 298–301.

$$\begin{array}{c} \text{PriestDist} \stackrel{\simeq}{\rightarrow} \text{FinSup}_{\text{DL}}^{\text{op}} \\ & & & \uparrow \\ \text{EsaDist} \stackrel{\simeq}{\rightarrow} \text{FinSup}_{\text{HA}}^{\text{op}} \\ & & & \uparrow \\ & & & & \uparrow \\ \text{BooRel} \stackrel{\simeq}{\rightarrow} \text{FinSup}_{\text{BA}}^{\text{op}} \end{array}$$

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 $\mathsf{PriestDist} \xrightarrow{\simeq} \mathsf{FinSup}_{\mathrm{DL}}^{\mathrm{op}}$ $\begin{array}{c} & & & \\ & & & \\$

Theorem

1. For X in **Priest**, the following assertions are equivalent.

(i) $i: X_p \to X, x \mapsto x$ is down-wards open. (ii) $i_*: X_p \to X$ has a right adjoint (necessarily given by i^*). (iii) X is a split subobject of a Boolean space in **PriestDist**.

 $PriestDist \xrightarrow{\simeq} FinSup_{DL}^{op}$ $\begin{array}{c} & & & & \\ & & & \\ & &$

Theorem

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- 2. PriestDist is idempotent split complete.

$$\begin{array}{c} \text{PriestDist} \xrightarrow{\simeq} \text{FinSup}_{DL}^{op} \\ & & & & & \\ & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & &$$

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- 1. For X in Priest, the following assertions are equivalent.
 - (i) $i: X_p \to X, x \mapsto x$ is down-wards open. (ii) $i_*: X_p \to X$ has a right adjoint (necessarily given by i^*). (iii) X is a split subobject of a Boolean space in **PriestDist**.
- 2. **PriestDist** is idempotent split complete.

$$\begin{array}{c} \text{PriestDist} \xrightarrow{\simeq} \text{FinSup}_{DL}^{\mathrm{op}} \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ \text{kar}(\text{BooRel}) \simeq \text{EsaDist} \xrightarrow{\simeq} \text{FinSup}_{HA}^{\mathrm{op}} \simeq \text{kar}(\text{FinSup}_{BA})^{\mathrm{op}} \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & &$$

Theorem

1. For X in **Priest**, the following assertions are equivalent.

(i) i: $X_p \to X$, $x \mapsto x$ is down-wards open.

- (ii) $i_*: X_p \leftrightarrow X$ has a right adjoint (necessarily given by i^*).
- (iii) X is a split subobject of a Boolean space in **PriestDist**.
- 2. **PriestDist** is idempotent split complete.

J. C. C. McKinsey and Alfred Tarski (1946). "On closed elements in closure algebras". In: *Annals of Mathematics. Second Series* **47**.(1), pp. 122–162.

Theorem

- 1. For X in **PosComp**, the following assertions are equivalent.
 - (i) $i: X_p \to X, x \mapsto x$ is down-wards open. (ii) $i_*: X_p \to X$ has a right adjoint (necessarily given by i^*). (iii) X is a split subobject of a comp. Hausdorff space in **PosComp**_V.
- 2. **PriestDist** is idempotent split complete.

$$\begin{array}{c} \text{PriestDist} \xrightarrow{\simeq} \text{FinSup}_{DL}^{\mathrm{op}} \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ \text{kar}(\text{BooRel}) \simeq \text{EsaDist} \xrightarrow{\simeq} \text{FinSup}_{HA}^{\mathrm{op}} \simeq \text{kar}(\text{FinSup}_{BA})^{\mathrm{op}} \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & &$$

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(i) i: $X_p \to X$, $x \mapsto x$ is down-wards open.

- (ii) $i_*: X_p \leftrightarrow X$ has a right adjoint (necessarily given by i^*).
- (iii) X is a split subobject of a comp. Hausdorff space in $\mathbf{PosComp}_{\mathbb{V}}$.
- 2. **PosComp**_{\mathbb{V}} is idempotent split complete.^a

^aAchim Jung, Mathias Kegelmann, and M. Andrew Moshier (2001). "Stably compact spaces and closed relations". In: *Electronic Notes in Theoretical Computer Science* **45**, pp. 209–231.

Theorem

- 1. For X in \mathcal{U} -Rep, the following assertions are equivalent.
 - (i) $i: X_p \to X, x \mapsto x$ is down-wards open. (ii) $i_*: X_p \to X$ has a right adjoint (necessarily given by i^*). (iii) X is a split subobject of a comp. Hausdorff space in $(U-\mathbf{Rep})_{\mathbb{V}}$.
- 2. **PosComp** $_{\mathbb{W}}$ is idempotent split complete.

$$\begin{array}{c} \text{PriestDist} \xrightarrow{\simeq} \text{FinSup}_{DL}^{\mathrm{op}} \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ \text{kar}(\text{BooRel}) \simeq \text{EsaDist} \xrightarrow{\simeq} \text{FinSup}_{HA}^{\mathrm{op}} \simeq \text{kar}(\text{FinSup}_{BA})^{\mathrm{op}} \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & &$$

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(i) i: $X_p \to X, x \mapsto x$ is down-wards open.

- (ii) $i_*: X_p \leftrightarrow X$ has a right adjoint (necessarily given by i^*).
- (iii) X is a split subobject of a comp. Hausdorff space in $(\mathfrak{U}$ -**Rep**)_{\mathbb{W}}.
- 2. **PosComp** $_{\mathbb{V}}$ is idempotent split complete.

Question

I do not know if $(\mathcal{U}$ -**Rep**)_{w} is idempotent split complete.