Duality theory, convergence, and enriched categories

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Collaboration with

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- Carla Reis
- Lurdes Sousa
- Isar Stubbe
- Walter Tholen

Motivation (part I)

F. William Lawvere (1973). "Metric spaces, generalized logic, and closed categories". In: *Rendiconti del Seminario Matemàtico e Fisico di Milano* **43**.(1), pp. 135–166. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.

"While listening to a 1967 lecture of Richard Swan, which included a discussion of the relative codimension of pairs of subvarieties, I noticed the analogy between the triangle inequality and a categorical composition law.

Ordered sets and metric spaces

$$\begin{array}{ll} \top \implies (x \leq x), & (x \leq y \;\&\; y \leq z) \implies x \leq z \\ 0 \geqslant d(x,x), & d(x,y) + d(y,z) \geqslant d(x,z) \end{array}$$

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"While listening to a 1967 lecture of Richard Swan, which included a discussion of the relative codimension of pairs of subvarieties, I noticed the analogy between the triangle inequality and a categorical composition law. Later I saw that Hausdorff had mentioned the analogy between metric spaces and posets."

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$$0 \geqslant d(x,x), \qquad d(x,y) + d(y,z) \geqslant d(x,z)$$

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Felix Hausdorff (1914). *Grundzüge der Mengenlehre*. Leipzig: Veit & Comp. VIII and 476 pages.

Thinking of an order relation on a set M as a function

$$f : M \times M \longrightarrow \{<, >, =\},$$

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$$f : M \times M \longrightarrow \{<, >, =\},$$

Hausdorff observes that

> *"Nun steht einer Verallgemeinerung dieser Vorstellung nichts im Wege, und wir können uns denken, daß eine beliebige Funktion der Paare einer Menge definiert, d.h. jedem Paar (a, b) von Elementen einer Menge M ein bestimmtes Element $n = f(a, b)$ einer zweiten Menge N zugeordnet sei. In noch weiterer Verallgemeinerung können wir eine Funktion der Elementtripel, Elementfolgen, Elementkomplexe, Teilmengen u. dgl. von M in Betracht ziehen."*

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Thinking of an order relation on a set M as a function

$$f : M \times M \longrightarrow \{<, >, =\},$$

Hausdorff observes that

> *Now there stands nothing in the way of a generalisation of this idea, and we can think of an arbitrary function of pairs of points which associates to each pair (a, b) of elements of a set M a specific element $n = f(a, b)$ of a second set N. Generalising further, we can consider a function of triples, sequences, complexes, subsets, etc.*

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... but also writes:

"Eine ganz allgemein gehaltene Theorie dieser Art würde natürlich erhebliche Komplikationen bedingen und wenig positive Ausbeute liefern."

A very general theory of this kind would bring considerable complications and little benefit.

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Hausdorff might think of a structure like $TX \times X \longrightarrow \mathcal{V}$...

...where $\mathbb{T} = (T, m, e)$ is a monad and $\mathcal{V} = (\mathcal{V}, \otimes, k)$ is a quantale!?

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Example

- A topology on X is a relation $a\colon UX \times X \longrightarrow \mathbf{2}$ so that

$$\top \implies (\dot{x} \xrightarrow{a} x) \quad \text{and} \quad (\mathfrak{X} \xrightarrow{Ua} \mathfrak{x} \,\&\, \mathfrak{x} \xrightarrow{a} x) \implies m_X(\mathfrak{X}) \xrightarrow{a} x$$

$$1_X \leq a \cdot e_X \quad \text{and} \quad a \cdot Ua \leq a \cdot m_X \;\; (\mathit{Note}: e_X \dashv e_X^\circ,\; m_X \dashv m_X^\circ).$$

Michael Barr (1970). “Relational algebras”. In: *Reports of the Midwest Category Seminar, IV*. Lecture Notes in Mathematics, Vol. 137. Springer, Berlin, pp. 39–55.

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- A topology on X is a map $a\colon X \to FX$ so that $e_X \leq a$ and $a \circ a \leq a$.

Werner Gähler (1992). "Monadic topology – A new concept of generalized topology". In: *Recent developments of general topology and its applications.* Berlin: Akademie-Verlag, pp. 136–149. International conference in memory of Felix Hausdorff (1868 - 1942), held in Berlin, Germany, March 22-28, 1992.

Motivation (part I)

Definition

A topological theory $\mathcal{T} = (\mathbb{T}, \mathcal{V}, \xi)$ consists of a monad $\mathbb{T} = (T, m, e)$, a quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$ and a map $\xi\colon T\mathcal{V} \to \mathcal{V}$ so that

- T preserves weak pullbacks and each naturality square of m is a weak pullback;
- $\xi\colon T\mathcal{V} \to \mathcal{V}$ is the structure of a lax Eilenberg–Moore algebra on $\mathcal{V}$ and ξ is "compatible with suprema in $\mathcal{V}$";
- the monoid operations are lax homomorphisms:

$$
\begin{tikzcd}
T1 \arrow[r, "Tk"] \arrow[d] \arrow[dr, phantom, "\leq"] & T\mathcal{V} \arrow[d, "\xi"] \\
1 \arrow[r, "k"'] & \mathcal{V}
\end{tikzcd}
\qquad
\begin{tikzcd}
T(\mathcal{V} \times \mathcal{V}) \arrow[r, "T\otimes"] \arrow[d] \arrow[dr, phantom, "\leq"] & T\mathcal{V} \arrow[d, "\xi"] \\
\mathcal{V} \times \mathcal{V} \arrow[r, "\otimes"'] & \mathcal{V}
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Dirk Hofmann (2007). "Topological theories and closed objects". In: *Advances in Mathematics* **215**.(2), pp. 789–824.

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Walter Tholen (2016). *Lax Distributive Laws for Topology, I.* Tech. rep. arXiv: 1603.06251 [math.CT].

. . . And Now for Something Completely Different!

Motivation (part II)

Duality theory (discrete case)

$$\mathsf{CoAlg}(V) \simeq \mathbf{BAO}^{\mathrm{op}}$$

Here $V : \mathbf{BooSp} \longrightarrow \mathbf{BooSp}$ is the Vietoris functor.

Bjarni Jónsson and Alfred Tarski (1951). "Boolean algebras with operators. I". In: *American Journal of Mathematics* **73**.(4), pp. 891–939.

Paul R. Halmos (1956). "Algebraic logic I. Monadic Boolean algebras". In: *Compositio Mathematica* **12**, pp. 217–249.

Clemens Kupke, Alexander Kurz, and Yde Venema (2004). "Stone coalgebras". In: *Theoretical Computer Science* **327**.(1-2), pp. 109–134.

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$$\begin{tikzcd} \mathbf{BooSp} \arrow[r, "\mathsf{hom}(-,\mathbf{2})", "\simeq"'] \arrow[d] & \mathbf{BA}^{\mathrm{op}} \arrow[d] \\ \mathbf{BooSp}_{\mathbb{V}} \arrow[r, "\simeq", "\mathsf{hom}(-,1)"'] & \mathbf{FinSup}^{\mathrm{op}}_{\mathrm{BA}} \end{tikzcd} \quad \rightsquigarrow$$

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Marshall Harvey Stone (1936). “The theory of representations for Boolean algebras”. In: *Transactions of the American Mathematical Society* **40**.(1), pp. 37–111.

Motivation (part II)

Duality theory (ordered case)

$$\begin{tikzcd} \mathbf{Priest} \arrow[r, "\mathrm{hom}(-{,}\mathbf{2})", "\simeq"'] \arrow[d] & \mathbf{DL}^{\mathrm{op}} \arrow[d] \\ \mathbf{Priest}_{\mathbb{V}} \arrow[r, "\mathrm{hom}(-{,}1)"'] & \mathbf{FinSup}^{\mathrm{op}} \end{tikzcd}$$

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Hilary A. Priestley (1970). "Representation of distributive lattices by means of ordered stone spaces". In: *Bulletin of the London Mathematical Society* **2**.(2), pp. 186–190.

Roberto Cignoli, S. Lafalce, and Alejandro Petrovich (1991). "Remarks on Priestley duality for distributive lattices". In: *Order* **8**.(3), pp. 299–315.

Alejandro Petrovich (1996). "Distributive lattices with an operator". In: *Studia Logica* **56**.(1-2), pp. 205–224. Special issue on Priestley duality.

Marcello M. Bonsangue, Alexander Kurz, and Ingrid M. Rewitzky (2007). "Coalgebraic representations of distributive lattices with operators". In: *Topology and its Applications* **154**.(4), pp. 778–791.

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These results suggest:

We are looking for a(n at least) fully faithful functor

$$\mathbf{PosComp}_{\mathbb{V}} \xrightarrow{\text{“}C=\hom(-,[0,1])\text{”}} (???)^{\mathrm{op}}$$

where (???) is the category of (for $([0,1], \otimes, 1)$ with $\otimes$ is continuous)

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- with monoid structure $\otimes$ and neutral element 1, laxly preserved.

Quantales

Examples

1. $\mathcal{V} = \mathbf{2}$ with $\otimes = \&$ and $k = \top$: $\mathbf{2}$-$\mathbf{Cat} \simeq \mathbf{Ord}$.
2. $\mathcal{V} = \overleftarrow{[0,\infty]}_+$ with $\otimes = +$ and $k = 0$: $\overleftarrow{[0,\infty]}_+$-$\mathbf{Cat} \simeq \mathbf{Met}$.

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Karl Menger (1942). "Statistical metrics". In: *Proceedings of the National Academy of Sciences of the United States of America* **28**, pp. 535–537.

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   - tensor: $f \otimes g(\gamma) = \bigvee_{\alpha+\beta\leqslant\gamma} f(\alpha) * g(\beta)$,
   - neutral element: $f_{0,1}(0) = 0$, $f_{0,1}(\alpha) = 1$ for $\alpha > 0$.

Vietoris monads

The discrete case

The functor $V\colon \mathbf{CompHaus} \longrightarrow \mathbf{CompHaus}$ is defined by

Leopold Vietoris (1922). "Bereiche zweiter Ordnung". In: *Monatshefte für Mathematik und Physik* **32**.(1), pp. 258–280.

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The ordered case

We consider here $V\colon \mathbf{PosComp} \longrightarrow \mathbf{PosComp}$ defined by

Leopoldo Nachbin (1950). *Topologia e Ordem*. Univ. of Chicago Press.

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Leopoldo Nachbin (1950). *Topologia e Ordem*. Univ. of Chicago Press.

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Duality theory for spectral distributors

We consider:

$$\begin{tikzcd}
\mathbf{PosComp}_{\mathbb{V}} \arrow[rr, "C"] & & \mathsf{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}, \\
& \mathbf{PosComp} \arrow[ul] \arrow[ur, "C=\hom(-,[0,1]^{\mathrm{op}})"'] &
\end{tikzcd}$$

The induced monad morphism j is precisely given by the family of maps

$$j_X\colon VX \longrightarrow [CX,[0,1]],\; A \longmapsto \Phi_A,$$

with $\Phi_A\colon CX \longrightarrow [0,1],\; \psi \longmapsto \sup_{x\in A} \psi(x)$.

Dirk Hofmann and Pedro Nora (2016). *Enriched Stone-type dualities.* Tech. rep. arXiv: 1605.00081 [math.CT].

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The case of a general tensor

Remark

For $\otimes = *$ or $\otimes = \odot$: $C\colon \mathbf{PosComp}_{\mathbb{V}} \to [0,1]\text{-}\mathbf{FinSup}^{\mathrm{op}}$ is not full.[a]

[a] Wait till 09h43$\pm\varepsilon$.

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  Here we add the operations $(\varphi \ominus u)(x) = \varphi(x) \ominus u \qquad (\varphi\colon X \to [0,1])$.

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$$\mathsf{Mon}([0,1]\text{-}\mathbf{FinSup})\ (\text{if } \otimes = \odot) \quad \text{resp.} \quad \mathsf{Mon}_{\ominus}([0,1]\text{-}\mathbf{FinSup})\ (\text{if } \otimes = *)$$

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We say that an object A of $\mathbf{A}$ has enough characters whenever the cone $(\varphi : A \to [0,1])_{\varphi}$ of all morphisms into $[0,1]$ separates the points of A.

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For $\otimes = *$ multiplication resp. $\otimes = \odot$ the Łukasiewicz tensor,

$$\mathbf{PosComp}_{\mathbb{V}}^{\mathrm{op}} \simeq \mathbf{A}_{[0,1],\mathrm{cc},\mathrm{lax}} \qquad \textit{and} \qquad \mathbf{PosComp}^{\mathrm{op}} \simeq \mathbf{A}_{[0,1],\mathrm{cc}}.$$

Next goals

Recall

We considered

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- We obtain the categories $\mathcal{T}$-**Cat** and (for $\mathcal{T}$ strict) $\mathcal{T}$-**Dist**.

Examples

- $\mathbb{T} = \mathbb{1}$, $\xi = 1 \colon \mathcal{V} \to \mathcal{V}$: $\mathcal{T}$-**Cat** $=$ $\mathcal{V}$-**Cat**.
- $\mathbb{T} = \mathbb{U}$, $\xi \colon U\mathbf{2} \to \mathbf{2} =$ "identity": $\mathcal{T}$-**Cat** $=$ **Top**.
- More general, for $\mathcal{V}$ compl. distr.: $\mathbb{T} = \mathbb{U}$, $\xi \colon U\mathcal{V} \to \mathcal{V}$, $\mathfrak{v} \mapsto \bigvee \bigwedge \mathfrak{v}$.
- $\mathbb{T} = \mathbb{W}$ free monoid monad, $\xi \colon W\mathcal{V} \to \mathcal{V}, (v_1, \dots, v_n) \mapsto v_1 \otimes \dots \otimes v_n$: Here $\mathcal{T}$-**Cat** $=$ $\mathcal{V}$-**MultiCat**; for $\mathcal{V} = \mathbf{2}$, $\mathcal{T}$-**Cat** $=$ **MultiOrd**.

Motivation

$$\mathbf{OrdCH} \to \mathbf{Top}$$
$$(X, \leq, \tau) \mapsto (X, {\downarrow}\tau)$$

Some facts

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$$\iff$$

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Representable $\mathcal{T}$-categories

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Motivation

$$\begin{tikzcd} \mathbf{Ord}^{\mathbb{U}} \simeq \mathbf{OrdCH} \arrow[r] & \mathbf{Top} \arrow[loop above, "\mathbb{U}\text{ of KZ-type}"] \end{tikzcd}$$

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Dimitri Chikhladze, Maria Manuel Clementino, and Dirk Hofmann (2015). "Representable $(\mathbb{T}, \mathcal{V})$-categories". In: *Applied Categorical Structures* **23**.(6), pp. 829–858.

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- Under some conditions: Representable $\implies$ Cauchy-complete.

On cocomplete $\mathcal{T}$-categories

For $\mathcal{T} = (\mathbb{T}, \mathcal{V}, \xi)$ strict:

- $a : (TX)^{\mathrm{op}} \otimes X \longrightarrow \mathcal{V}$ is a $\mathcal{V}$-functor and there is an adjunction

$$\begin{tikzcd} \mathcal{T}\text{-}\mathbf{Dist}^{\mathrm{op}} \arrow[r, bend left, "P"] \arrow[r, phantom, "\top"] & \mathcal{T}\text{-}\mathbf{Cat} \arrow[l, bend left, "(-)^{\circledast}"] \arrow[loop right, "\mathbb{P} \text{ of KZ-type}"] \end{tikzcd}$$

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Examples

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[a] Dana Scott (1972). "Continuous lattices". In: *Toposes, algebraic geometry and logic (Conf., Dalhousie Univ., Halifax, N. S., 1971)*. Vol. 274. Springer, Lect. Notes Math., pp. 97–136.

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- For **MultiOrd**: $\mathbf{MultiOrd}^{\mathbb{P}} \simeq \mathbf{Set}^{\mathbb{P}} \simeq \mathbf{Quantales}$[a].

[a] Joachim Lambek, Michael Barr, John F. Kennison, and Robert Raphael (2012). "Injective hulls of partially ordered monoids". In: *Theory and Applications of Categories* **26**.(13), pp. 338–348.

The enriched Vietoris monad (assuming T1=1)

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Theorem (for a dualisable $\mathcal{T}$-category X)

X^{op} is a $\mathcal{T}$-category $\iff$ X is core-compact[a] $\iff$ X is representable.

[a] $a \cdot T_\xi a = a \cdot m_X$

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Theorem (For each $\mathcal{T}$-category X)

The $\mathcal{T}$-graph $\mathcal{V}^X$ is dualisable and $VX = (\mathcal{V}^X)^{\mathrm{op}}$ is a $\mathcal{T}$-category.

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- $VX = (\mathbf{2}^X)^{\mathrm{op}}$ is the lower Vietoris space.
- X exp. $\iff$ $(VX)^{\mathrm{op}}$ is top. $\iff$ VX is core-c. $\iff$ X is core-c.

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Theorem

*This construction defines a monad $\mathbb{V} = (V, m, e)$ of co-KZ-type on $\mathcal{T}$-**Cat** and $\mathcal{T}$-**Rep**.*

The enriched Vietoris monad (continuação)

Remark

- There is a functor $(-)_0 \colon \mathcal{T}\text{-}\mathbf{Rep}_{\mathbb{V}} \longrightarrow \mathcal{V}\text{-}\mathbf{Dist}$; moreover

$$X \overset{\varphi}{\multimap} Y = X_0 \overset{\varphi}{\multimap} Y_0 \quad + \quad \begin{tikzcd} T(X_0) \arrow[r, "T_\xi(\varphi)", "\circ" marking] \arrow[d, "a"', "\circ" marking] & T(Y_0) \arrow[d, "b", "\circ" marking] \\ X_0 \arrow[r, "\varphi"', "\circ" marking] & Y_0. \end{tikzcd}$$

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---

[a] $a \cdot Tf^\circ \cdot T_\xi b_0 \geq f^\circ \cdot b$

The enriched Vietoris monad (continuação)

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For a study of (co)-Kock–Zöberlein monads in Topology:

Margarida Carvalho and Lurdes Sousa (2015). "On Kan-injectivity of Locales and Spaces". In: *Applied Categorical Structures*, pp. 1–22.

Jiří Adámek, Lurdes Sousa, and Jiří Velebil (2015). "Kan injectivity in order-enriched categories". In: *Mathematical Structures in Computer Science* **25**.(1), pp. 6–45.

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Example (in **Top**)

Let $\mathbb{D} = (D, m, e)$ be a co-Kock–Zöberlein monad on **Top** where each component e_X of e is an embedding. Then $\mathbb{D}$ is a submonad of $\mathbb{V}$.

Now back to duality theory

We consider now $\mathcal{U} = (\mathbb{U}, [0,1], \xi)$ with $\otimes = *$ or $\otimes = \odot$. Then

$$\begin{tikzcd} \mathcal{U}\text{-}\mathbf{Rep}_{\mathbb{V}} \arrow[rr, "C=\hom(-{,}1)"] & & {[0,1]}\text{-}\mathbf{FinSup}^{\mathrm{op}} \\ & \mathcal{U}\text{-}\mathbf{Rep}_{\mathrm{sep}} \arrow[ul] \arrow[ur, "C=\hom(-{,}[0{,}1]^{\mathrm{op}})"'] & \end{tikzcd}$$

induces the monad morphism

$$j_X : VX \longrightarrow [CX, [0,1]],\ (\varphi : 1 \multimapdotinv X) \longmapsto (\psi \mapsto \psi \cdot \varphi).$$

Duality theory for representable $\mathcal{U}$-distributors

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Theorem

$[0,1]$ *is an initial (=regular) cogenerator in* **PosComp**.

Leopoldo Nachbin (1950). *Topologia e Ordem*. Univ. of Chicago Press.

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Definition

A separated representable $\mathcal{U}$-category X is called $[0,1]^{\mathrm{op}}$-cogenerated if the cone $(\psi : X \to [0,1]^{\mathrm{op}})_{\psi \in CX}$ is point-separating and initial.

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X is $[0,1]^{\mathrm{op}}$-cogenerated $\implies$ VX is $[0,1]^{\mathrm{op}}$-cogenerated.

Duality theory for representable $\mathcal{U}$-categories

Theorem

The functor

$$C\colon (\mathcal{U}\text{-}\mathbf{Rep}_{[0,1]^{\mathrm{op}}})_{\mathbb{V}} \longrightarrow [0,1]\text{-}\mathbf{FinSup}^{\mathrm{op}}$$

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*Let X be a partially ordered compact space and $\psi_0 \in CX$. Then $\psi_0 \otimes -\colon CX \longrightarrow CX$ is the largest morphism $\Phi\colon CX \longrightarrow CX$ in $[0,1]$-**FinSup** satisfying*

$$\Phi(1) \leq \psi_0 \qquad \text{and} \qquad \Phi(\psi) \leq \psi \qquad (\text{for all } \psi \in CX).$$

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Corollary

A morphism $\varphi\colon X \mathrel{-\!\!\circ\!\!\rightarrow} Y$ in $\mathcal{U}$-$\mathbf{Rep}_{\mathbb{V}}$ between partially ordered compact spaces is in $\mathbf{PosComp}_{\mathbb{V}}$ if and only if $C\varphi$ preserves laxly the tensor.

Restricting to functors

Before

For $C\colon \mathbf{StablyComp}_{\mathbb{V}} \longrightarrow \mathrm{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$ and

$e_X\colon X \longrightarrow VX,\ x \longmapsto \overline{\{x\}} = \uparrow x$:

- $A \subseteq X$ closed $\quad\leftrightsquigarrow\quad \Phi : CX \longrightarrow [0,1]$.

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Cauchy complete $\mathcal{T}$-categories

$\mathcal{T}$-distributors

Recall: $\mathcal{T}$-distributor $\varphi : X \,{\circ\!\!\rightharpoonup}\, Y \;=\; \varphi\colon TX \times Y \to \mathcal{V}$ so that . . . .

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Definition

X is Cauchy complete if every adjunction $\varphi \dashv \psi$ is induced by some $x \in X$.

Maria Manuel Clementino and Dirk Hofmann (2009). "Lawvere completeness in topology". In: *Applied Categorical Structures* **17**.(2), pp. 175–210.

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Examples

- In **Met**: Cauchy complete = Cauchy complete[a].

[a] F. William Lawvere (1973). "Metric spaces, generalized logic, and closed categories". In: *Rendiconti del Seminario Matemàtico e Fisico di Milano* **43**.(1), pp. 135–166.

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- In **App** $\simeq \mathcal{U}_{\overleftarrow{[0,\infty]}_+}$-**Cat** $\simeq \mathcal{U}_{[0,1]_*}$-**Cat**: Cauchy complete = sober[a].

[a] Bernhard Banaschewski, Robert Lowen, and Cristophe Van Olmen (2006). "Sober approach spaces". In: *Topology and its Applications* **153**.(16), pp. 3059–3070.

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Proposition

Every representable $\mathcal{U}$-category is Cauchy complete.

Adjoint $\mathcal{T}$-distributors

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An $\mathcal{U}$*-distributor* $\varphi\colon 1 \,{\circ\!\!\!\rightharpoondown}\, X$ *is left adjoint* $\iff$ *the* $[0,1]$*-functor* $[\varphi,-]\colon \mathcal{U}\text{-}\mathbf{Cat}(X,[0,1]) \longrightarrow [0,1]$ *preserves tensors and finite suprema.*

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Leopoldo Nachbin (1992). "Compact unions of closed subsets are closed and compact intersections of open subsets are open". In: *Portugaliæ Mathematica* **49**.(4), pp. 403–409.

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For Łukasiewicz $\otimes = \odot$

$[0,1]$ is a Girard quantale: for every $u \in [0,1]$, $u = \hom(\hom(u,\bot),\bot)$ where $\hom(u,\bot) = 1-u =: u^{\perp}$.

Furthermore, the diagram

$$\begin{tikzcd}
{[0,1]\text{-}\mathbf{Dist}(X,1)} \arrow[r, "(-)^{\perp}"] \arrow[d, "(-\cdot\varphi)"'] & {[0,1]\text{-}\mathbf{Dist}(1,X)^{\mathrm{op}}} \arrow[d, "{[\varphi,-]^{\mathrm{op}}}"] \\
{[0,1]} \arrow[r, "(-)^{\perp}"'] & {[0,1]^{\mathrm{op}}}
\end{tikzcd}$$

commutes in $[0,1]$-$\mathbf{Cat}$

Adjoint $\mathcal{T}$-distributors

Proposition

An $\mathcal{U}$-distributor $\varphi\colon 1 \,{\circ\!\!\!\rightharpoondown}\, X$ is left adjoint $\iff$ the $[0,1]$-functor $[\varphi,-]\colon \mathcal{U}\text{-}\mathbf{Cat}(X,[0,1]) \longrightarrow [0,1]$ preserves tensors and finite suprema.

For Łukasiewicz $\otimes = \odot$

$[0,1]$ is a Girard quantale: for every $u \in [0,1]$, $u = \hom(\hom(u,\bot),\bot)$ where $\hom(u,\bot) = 1-u =: u^{\perp}$.

Furthermore, the diagram

$$\begin{tikzcd}
\mathcal{U}\text{-}\mathbf{Cat}(X,[0,1]^{\mathrm{op}}) \arrow[r, "(-)^{\perp}"] \arrow[d, "(-\cdot\varphi)"'] & \mathcal{U}\text{-}\mathbf{Cat}(X,[0,1])^{\mathrm{op}} \arrow[d, "{[\varphi,-]^{\mathrm{op}}}"] \\
{[0,1]} \arrow[r, "(-)^{\perp}"'] & {[0,1]^{\mathrm{op}}}
\end{tikzcd}$$

commutes in $[0,1]$-$\mathbf{Cat}$

Adjoint $\mathcal{T}$-distributors

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An $\mathcal{U}$-distributor $\varphi\colon 1 \mathrel{\circ\!\!\rightharpoonup} X$ is left adjoint $\iff$ the $[0,1]$-functor $[\varphi,-]\colon \mathcal{U}\text{-}\mathbf{Cat}(X,[0,1]) \longrightarrow [0,1]$ preserves tensors and finite suprema.

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$$\begin{tikzcd} CX \arrow[r, hook] \arrow[dr, "\Phi"'] & \mathcal{U}\text{-}\mathbf{Cat}(X,[0,1]^{\mathrm{op}}) \arrow[r, "(-)^\perp"] \arrow[d, "(-\cdot\varphi)"'] & \mathcal{U}\text{-}\mathbf{Cat}(X,[0,1])^{\mathrm{op}} \arrow[d, "{[\varphi,-]^{\mathrm{op}}}"] \\ & {[0,1]} \arrow[r, "(-)^\perp"'] & {[0,1]^{\mathrm{op}}} \end{tikzcd}$$

commutes in $[0,1]$-$\mathbf{Cat}$ and $CX \hookrightarrow \mathcal{U}\text{-}\mathbf{Cat}(X,[0,1]^{\mathrm{op}})$ is $\bigvee$-dense.

Adjoint $\mathcal{T}$-distributors

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commutes in $[0,1]$-$\mathbf{Cat}$ and $CX \hookrightarrow \mathcal{U}\text{-}\mathbf{Cat}(X,[0,1]^{\mathrm{op}})$ is $\bigvee$-dense.

Conclusion: $\varphi\colon 1 \,{\circ\!\!\rightharpoonup}\, X$ is left adjoint $\iff$ Φ preserves finite w. limits.

Esakia duality

$$\mathbf{Priest} \xrightarrow{\simeq} \mathbf{DL}^{\mathrm{op}}$$

$$\mathbf{BooSp} \xrightarrow{\simeq} \mathbf{BA}^{\mathrm{op}}$$

Esakia duality

$$\begin{tikzcd}
\mathbf{Priest} \arrow[r, "\simeq"] & \mathbf{DL}^{\mathrm{op}} \\
\mathbf{Esa} \arrow[u, hook] \arrow[r, "\simeq"] & \mathbf{HA}^{\mathrm{op}} \arrow[u, hook] \\
\mathbf{BooSp} \arrow[u, hook] \arrow[r, "\simeq"] & \mathbf{BA}^{\mathrm{op}} \arrow[u, hook]
\end{tikzcd}$$

Definition

A Priestley space $(X, \leq, \alpha)$ is called an Esakia space whenever $i\colon (X, \alpha) \to (X, \leq, \alpha)$ is downwards open.

Leo Esakia (1974). "Topological Kripke models". In: *Doklady Akademii Nauk SSSR* **214**, pp. 298–301.

Esakia duality

$$\begin{tikzcd}
\mathbf{PriestDist} \arrow[r, "\simeq"] & \mathbf{FinSup}^{\mathrm{op}}_{\mathrm{DL}} \\
\mathbf{EsaDist} \arrow[r, "\simeq"] \arrow[u, hook] & \mathbf{FinSup}^{\mathrm{op}}_{\mathrm{HA}} \arrow[u, hook] \\
\mathbf{BooRel} \arrow[r, "\simeq"] \arrow[u, hook] & \mathbf{FinSup}^{\mathrm{op}}_{\mathrm{BA}} \arrow[u, hook]
\end{tikzcd}$$

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\end{tikzcd}$$

Theorem

1. *For X in* **Priest**, *the following assertions are equivalent.*
   - *(i)* $i \colon X_p \to X$, $x \mapsto x$ *is down-wards open.*
   - *(ii)* $i_* \colon X_p \nrightarrow X$ *has a right adjoint (necessarily given by* i^**).*
   - *(iii)* X *is a split subobject of a Boolean space in* **PriestDist**.

Esakia duality

$$\begin{tikzcd}
\mathbf{PriestDist} \arrow[r, "\simeq"] & \mathbf{FinSup}_{\mathrm{DL}}^{\mathrm{op}} \\
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1. *For X in* **Priest**, *the following assertions are equivalent.*
   - *(i) $i \colon X_p \to X,\ x \mapsto x$ is down-wards open.*
   - *(ii) $i_* \colon X_p \mathrel{\circ\!\!\!\rightarrow} X$ has a right adjoint (necessarily given by i^*).*
   - *(iii) X is a split subobject of a Boolean space in* **PriestDist**.
2. **PriestDist** *is idempotent split complete.*

Esakia duality

$$\begin{tikzcd}
& \mathbf{PriestDist} \arrow[r, "\simeq"] & \mathbf{FinSup}_{\mathrm{DL}}^{\mathrm{op}} \\
\mathsf{kar}(\mathbf{BooRel}) \simeq & \mathbf{EsaDist} \arrow[r, "\simeq"] \arrow[u, hook] & \mathbf{FinSup}_{\mathrm{HA}}^{\mathrm{op}} \arrow[u, hook] \\
& \mathbf{BooRel} \arrow[r, "\simeq"] \arrow[u, hook] & \mathbf{FinSup}_{\mathrm{BA}}^{\mathrm{op}} \arrow[u, hook]
\end{tikzcd}$$

Theorem

1. *For X in* **Priest**, *the following assertions are equivalent.*
   (i) *$i\colon X_p \to X, x \mapsto x$ is down-wards open.*
   (ii) *$i_*\colon X_p \nrightarrow X$ has a right adjoint (necessarily given by i^*).*
   (iii) *X is a split subobject of a Boolean space in* **PriestDist**.
2. **PriestDist** *is idempotent split complete.*

Esaka duality

$$\begin{tikzcd}
 & \mathbf{PriestDist} \arrow[r, "\simeq"] & \mathbf{FinSup}_{\mathrm{DL}}^{\mathrm{op}} & \\
\mathsf{kar}(\mathbf{BooRel}) \simeq & \mathbf{EsaDist} \arrow[u, hook] \arrow[r, "\simeq"] & \mathbf{FinSup}_{\mathrm{HA}}^{\mathrm{op}} \arrow[u, hook] & \simeq \mathsf{kar}(\mathbf{FinSup}_{\mathrm{BA}})^{\mathrm{op}} \\
 & \mathbf{BooRel} \arrow[u, hook] \arrow[r, "\simeq"] & \mathbf{FinSup}_{\mathrm{BA}}^{\mathrm{op}} \arrow[u, hook] &
\end{tikzcd}$$

Theorem

1. *For X in* **Priest***, the following assertions are equivalent.*
   (i) $i\colon X_p \to X, x \mapsto x$ is down-wards open.
   (ii) $i_*\colon X_p \mathrel{\circ\!\!\rightarrow} X$ has a right adjoint (necessarily given by i^*).
   (iii) X is a split subobject of a Boolean space in **PriestDist***.*
2. **PriestDist** *is idempotent split complete.*

J. C. C. McKinsey and Alfred Tarski (1946). "On closed elements in closure algebras". In: *Annals of Mathematics. Second Series* **47**.(1), pp. 122–162.

Esakia duality

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\end{tikzcd}$$

Theorem

1. *For X in* **PosComp**, *the following assertions are equivalent.*
   (i) $i\colon X_p \to X,\ x \mapsto x$ *is down-wards open.*
   (ii) $i_*\colon X_p \nrightarrow X$ *has a right adjoint (necessarily given by i^*).*
   (iii) X *is a split subobject of a comp. Hausdorff space in* $\mathbf{PosComp}_{\mathbb{V}}$.
2. **PriestDist** *is idempotent split complete.*

Esakia duality

$$
\begin{tikzcd}
 & \mathbf{PriestDist} \arrow[r, "\simeq"] & \mathbf{FinSup}^{\mathrm{op}}_{\mathrm{DL}} & \\
\mathsf{kar}(\mathbf{BooRel}) \simeq & \mathbf{EsaDist} \arrow[u, hook] \arrow[r, "\simeq"] & \mathbf{FinSup}^{\mathrm{op}}_{\mathrm{HA}} \arrow[u, hook] & \simeq \mathsf{kar}(\mathbf{FinSup}_{\mathrm{BA}})^{\mathrm{op}} \\
 & \mathbf{BooRel} \arrow[u, hook] \arrow[r, "\simeq"] & \mathbf{FinSup}^{\mathrm{op}}_{\mathrm{BA}} \arrow[u, hook] &
\end{tikzcd}
$$

Theorem

1. *For X in* **PosComp**, *the following assertions are equivalent.*
   - *(i)* $i \colon X_p \to X, x \mapsto x$ *is down-wards open.*
   - *(ii)* $i_* \colon X_p \nrightarrow X$ *has a right adjoint (necessarily given by* i^**).*
   - *(iii)* X *is a split subobject of a comp. Hausdorff space in* $\mathbf{PosComp}_{\mathbb{V}}$.
2. $\mathbf{PosComp}_{\mathbb{V}}$ *is idempotent split complete.*[a]

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[a] Achim Jung, Mathias Kegelmann, and M. Andrew Moshier (2001). "Stably compact spaces and closed relations". In: *Electronic Notes in Theoretical Computer Science* **45**, pp. 209–231.

Esakia duality

$$\begin{tikzcd}
 & \mathbf{PriestDist} \arrow[r, "\simeq"] & \mathbf{FinSup}_{\mathrm{DL}}^{\mathrm{op}} & \\
\mathsf{kar}(\mathbf{BooRel}) \simeq & \mathbf{EsaDist} \arrow[r, "\simeq"] \arrow[u, hook] & \mathbf{FinSup}_{\mathrm{HA}}^{\mathrm{op}} \arrow[u, hook] & \simeq \mathsf{kar}(\mathbf{FinSup}_{\mathrm{BA}})^{\mathrm{op}} \\
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\end{tikzcd}$$

Theorem

1. *For X in $\mathcal{U}$-**Rep**, the following assertions are equivalent.*
   (i) *$i\colon X_p \to X$, $x \mapsto x$ is down-wards open.*
   (ii) *$i_*\colon X_p \nrightarrow X$ has a right adjoint (necessarily given by i^*).*
   (iii) *X is a split subobject of a comp. Hausdorff space in $(\mathcal{U}\text{-}\mathbf{Rep})_{\mathbb{V}}$.*
2. $\mathbf{PosComp}_{\mathbb{V}}$ *is idempotent split complete.*

Esakia duality

$$\begin{tikzcd}
 & \mathbf{PriestDist} \arrow[r, "\simeq"] & \mathbf{FinSup}_{\mathrm{DL}}^{\mathrm{op}} & \\
\mathsf{kar}(\mathbf{BooRel}) \simeq & \mathbf{EsaDist} \arrow[r, "\simeq"] \arrow[u, hook] & \mathbf{FinSup}_{\mathrm{HA}}^{\mathrm{op}} \arrow[u, hook] & \simeq \mathsf{kar}(\mathbf{FinSup}_{\mathrm{BA}})^{\mathrm{op}} \\
 & \mathbf{BooRel} \arrow[r, "\simeq"] \arrow[u, hook] & \mathbf{FinSup}_{\mathrm{BA}}^{\mathrm{op}} \arrow[u, hook] &
\end{tikzcd}$$

Theorem

1. *For X in $\mathcal{U}$-**Rep**, the following assertions are equivalent.*
   (i) *$i \colon X_p \to X, x \mapsto x$ is down-wards open.*
   (ii) *$i_* \colon X_p \nrightarrow X$ has a right adjoint (necessarily given by i^*).*
   (iii) *X is a split subobject of a comp. Hausdorff space in $(\mathcal{U}\text{-}\mathbf{Rep})_{\mathbb{V}}$.*
2. ***PosComp**$_{\mathbb{V}}$ is idempotent split complete.*

Question

I do not know if $(\mathcal{U}\text{-}\mathbf{Rep})_{\mathbb{V}}$ is idempotent split complete.