An embedding theorem for regular Mal'tsev categories

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Famous embedding theorems

Yoneda embedding

Every small category $\mathcal{C}$ admits a fully faithful embedding

$$\mathcal{C} \hookrightarrow \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$$

which preserves small limits.

Barr's embedding

Every small regular category $\mathcal{C}$ admits a fully faithful embedding

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which preserves finite limits and regular epimorphisms.

Lubkin's embedding

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Mal'tsev categories

Theorem (Carboni - Pedicchio - Pirovano, 1992)

The followings conditions on a finitely complete category $\mathcal{C}$ are equivalent:

1. any reflexive relation is an equivalence relation,
2. any reflexive relation is symmetric,
3. any reflexive relation is transitive,
4. every relation is difunctional.

In this case, we say that $\mathcal{C}$ is a Mal'tsev category.

Regular Mal'tsev categories

Theorem (Carboni - Lambek - Pedicchio, 1990)

The followings conditions on a regular category $\mathcal{C}$ are equivalent:

1. $\mathcal{C}$ is a Mal'tsev category,
2. for any pair (R, S) of equivalence relations on a same object, their composition RS is an equivalence relation,
3. for any pair (R, S) of equivalence relations on a same object, $RS = SR$.

Aim

Find a regular Mal'tsev category $\mathcal{M}$ such that, for every small regular Mal'tsev category $\mathcal{C}$, there exists a faithful conservative embedding

$$\mathcal{C} \hookrightarrow \mathcal{M}^{\mathcal{D}}$$

which preserves finite limits and regular epimorphisms.

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Definition

A *finitary essentially algebraic theory* is a quintuple $\Gamma = (S, \Sigma, E, \Sigma_t, \mathrm{Def})$ where

- S is a set of sorts,
- Σ is a set of S-sorted finitary operation symbols,
- E is a set of Σ-equations,
- $\Sigma_t \subseteq \Sigma$ is the subset of 'total operation symbols',
- for each $\sigma \in \Sigma \setminus \Sigma_t$, $\mathrm{Def}(\sigma)$ is a finite set of Σ_t-equations (in the variables of σ).

Definition

A Γ-*model* A is the collection of

- an S-sorted set $(A_s)_{s \in S} \in \mathrm{Set}^S$,
- for each $\sigma \colon s_1 \times \cdots \times s_n \to s$ in Σ, a partial function $\sigma^A \colon A_{s_1} \times \cdots \times A_{s_n} \to A_s$

satisfying

- for each $\sigma \in \Sigma_t$, σ^A is totally defined,
- for each $\sigma \in \Sigma \setminus \Sigma_t$, $\sigma(a_1, \ldots, a_n)$ is defined if and only if $(a_1, \ldots, a_n)$ satisfies the equations of $\mathrm{Def}(\sigma)$ in A,
- A satisfies the equations of E whenever they are defined.

This gives rise to the category $\mathrm{Mod}(\Gamma)$.

Theorem (Gabriel - Ulmer, 1971)

Up to equivalence, the categories of the form $\mathrm{Mod}(\Gamma)$ are exactly the locally finitely presentable categories.

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Example

The category Cat is of the form Mod(Γ):

- two sorts: O and A
- operations:

- $\mathrm{Def}(m) = \{(f,g) \in A^2 \mid c(f) = d(g)\}$
- + axioms.

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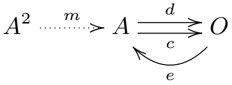

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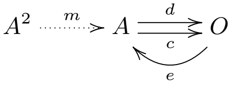

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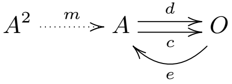

- $\mathrm{Def}(m) = \{(f,g) \in A^2 \mid c(f) = d(g)\}$
- $+$ axioms.

Theorem (Mal'tsev, 1954)

A variety of universal algebras $\mathbb{V}$ is a Mal'tsev category if and only if its theory contains a ternary operation $p(x, y, z)$ satisfying the identities

$$\begin{cases} p(x, y, y) = x \\ p(x, x, y) = y. \end{cases}$$

Theorem

Let Γ be an essentially algebraic theory. Then $\mathrm{Mod}(\Gamma)$ is a Mal'tsev category if and only if, for each sort $s \in S$, there exists in Γ a term $p^s \colon s^3 \to s$ such that

- $p^s(x, x, y)$ and $p^s(x, y, y)$ are everywhere-defined and
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The category $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})$

We construct a finitary essentially algebraic theory Γ_{Mal} such that:

- for each sort $s \in S_{\mathrm{Mal}}$, there exists a sort $\overline{s}$ and operation symbols

$$\begin{array}{ccc} s^3 & \xrightarrow{\rho^s} & \overline{s} \\ & \nearrow_{\alpha^s} & \\ s & & \end{array}$$

  satisfying the axioms

$$\begin{cases} \rho^s(x,y,y) = \alpha^s(x) \\ \rho^s(x,x,y) = \alpha^s(y) \\ \pi^s(\alpha^s(x)) = x \end{cases}$$

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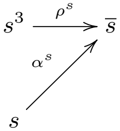

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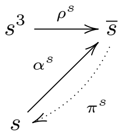

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Every small regular Mal'tsev category $\mathcal{C}$ admits a faithful conservative embedding

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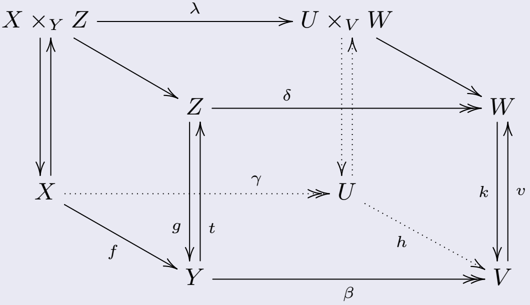
Application

Proposition (Bourn, 2003)

Let $\mathcal{C}$ be a regular Mal'tsev category. For any commutative diagram



if γ and δ are regular epimorphisms, then the comparison morphism λ is also a regular epimorphism.

The varietal proof

- Let $(u, w) \in U \times_V W$. So $u \in U$ and $w \in W$ are such that $h(u) = k(w)$.
- Since γ and δ are surjective, there exist $x \in X$ and $z \in Z$ such that $\gamma(x) = u$ and $\delta(z) = w$.
- Let $z' = p(z, tg(z), tf(x)) \in Z$.
- $(x, z') \in X \times_Y Z$ since

$$g(z') = p(g(z), gtg(z), gtf(x)) = p(g(z), g(z), f(x)) = f(x).$$

- $\lambda(x, z') = (u, w)$ since $\gamma(x) = u$ and

$$\begin{aligned} \delta(z') &= p(\delta(z), \delta tg(z), \delta tf(x)) = p(\delta(z), vk\delta(z), vh\gamma(x)) \\ &= p(w, vk(w), vh(u)) = p(w, vk(w), vk(w)) = w. \end{aligned}$$

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- $(u, w) = \pi^s(\alpha^s(u, w)) = \pi^s(\lambda(\alpha^s(x), z')) \in \text{Im}(\lambda)$.

Ingredients of the proof

The proof relies on the following key ingredients:

- Approximate Mal'tsev operations (Bourn - Janelidze, 2008).
- A $\mathcal{C}$-projective covering of $\mathrm{Lex}(\mathcal{C},\mathrm{Set})^{\mathrm{op}}$ (Grothendieck 1957, Barr 1986).
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Generalisation

We have similar embedding theorems for the following classes of categories:

- n-permutable categories,
- regular unital categories,
- regular strongly unital categories,
- regular subtractive categories,
- ...

Thank you for your attention!