# Towards a categorification of integers

Piotr Jędrzejewicz
Nicolaus Copernicus University
Toruń, Poland

CT 2017: International Category Theory Conference
University of British Columbia
Vancouver, BC, Canada
July 16 - 22, 2017

# Schanuel's question

"Where are negative sets?

Though ill-posed, the question is suggestive; a good answer should complete the diagram

$$\begin{array}{ccc} \mathbb{S} & \hookrightarrow & \mathbb{E} \\ \downarrow & & \downarrow \\ \mathbb{N} & \hookrightarrow & \mathbb{Z} \end{array}$$

where $\mathbb{S}$ is the category of finite sets; we seek an enlargement $\mathbb{E}$, the isomorphism classes of which should give rise to all integers, rather than just natural numbers."

# Schanuel's question

The answer is negative under some natural assumptions about products, coproducts, and the initial object.

Stephen H. Schanuel,
*Negative sets have Euler characteristic and dimension*,
*Category Theory, Como 1990*, Lecture Notes in Mathematics 1488, Springer, Berlin 1991, 379–385.

# The aim

We would like to present a background for constructing a positive answer to Schanuel's question provided we loose only the assumption about the initial object.

Multiset is a set with repeated elements.

Example – the multiset of prime factors of 360:

$$[2, 2, 2, 3, 3, 5].$$

Multisets were rediscovered for many times during the history of mathematics. For a survey see:

Wayne Blizard, *The development of multiset theory*, 1991, Modern Logic 1, 319 – 352.

# Generalized multisets

The first known observation that one can define a generalized multiset with arbitrary integer multiplicities, belongs to Hassler Whitney:

"Suppose we associate with each element of a set $R'$ any integer, positive, negative or zero, instead of merely one or zero. The resulting function will not in general be the characteristic function of a real set; but we may consider it as the characteristic function of a *generalized set*, where each element is counted any number of times."

Hassler Whitney, *Characteristic functions and the algebra of logic*, Annals of Mathematics 34 (1933), 405 – 414.

# Generalized multisets

Systematic studies in this field started with the works of Wolfgang Reisig, Wayne D. Blizard and Daniel Loeb.

Wolfgang Reisig, *Petri nets, An introduction*, Chapter 9, EATCS Monographs on Theoretical Compututer Science 4, Springer, Berlin 1985.

Wayne D. Blizard, *Negative membership*, Notre Dame Journal of Formal Logic 31 (1990), 346–368.

Daniel Loeb, *Sets with a negative number of elements*, Advances in Mathematics 91 (1992), 64 – 74.

# Example of a generalized multiset

We have:

$$\frac{360}{539} = \frac{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5}{7 \cdot 7 \cdot 11}.$$

The multiset of prime factors:

$$[2, 2, 2, 3, 3, 5 \,|\, 7, 7, 11].$$

# Definitions

A **multiset** in the space $U$ is defined by its multiplicity function:

$$\nu\colon U \to \{0, 1, 2, 3, \dots\}.$$

A **generalized multiset** is defined by a function:

$$\nu\colon U \to \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

# Definition

When we restrict multiplicities to:

$$1, 0, -1,$$

we obtain a **generalized set** $X$ which is a pair of disjoint sets $(A, B)$, where $A$ is the positive part and $B$ is the negative one:

$$A = \{z \in U : \nu_X(z) = 1\},$$

$$B = \{z \in U : \nu_X(z) = -1\}.$$

# Definition

If $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$, then we write

$$X = \{a_1, \ldots, a_m \mid b_1, \ldots, b_n\}.$$

The generalized number of elements:

$$|X|_g = m - n.$$

# Inclusion

Inclusion of generalized sets $X \subset_g Y$ is defined by:

$$\forall_{z \in U}\ \nu_X(z) \leqslant \nu_Y(z).$$

Equivalently:

$$\begin{cases} \forall_{z \in U}\ (\nu_X(z) = 1 \Rightarrow \nu_Y(z) = 1) \\ \forall_{z \in U}\ (\nu_Y(z) = -1 \Rightarrow \nu_X(z) = -1). \end{cases}$$

# Inclusion

$$\begin{cases} \forall_{z\in U} \left(\nu_X(z) = 1 \Rightarrow \nu_Y(z) = 1\right) \\ \forall_{z\in U} \left(\nu_Y(z) = -1 \Rightarrow \nu_X(z) = -1\right) \end{cases}$$

Given generalized sets

$$X = (A, B), \quad Y = (C, D),$$

where $A \cap B = \varnothing$, $C \cap D = \varnothing$, then:

$$(A, B) \subset_g (C, D) \Leftrightarrow A \subset C \wedge D \subset B.$$

# Intersection

Intersection of generalized sets $X$, $Y$:

$$\nu_{X\cap_g Y}(z) = \min(\nu_X(z), \nu_Y(z)), \quad z \in U.$$

We have a table:

| $\nu_X(z)$ \ $\nu_Y(z)$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $-1$ | $-1$ | $-1$ | $-1$ |
| $0$ | $-1$ | $0$ | $0$ |
| $1$ | $-1$ | $0$ | $1$ |

# Intersection

| $\nu_X(z)$ \ $\nu_Y(z)$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $-1$ | $-1$ | $-1$ | $-1$ |
| $0$ | $-1$ | $0$ | $0$ |
| $1$ | $-1$ | $0$ | $1$ |

Given $X = (A, B)$, $Y = (C, D)$, where $A \cap B = \varnothing$, $C \cap D = \varnothing$, then:

$$(A, B) \cap_g (C, D) = (A \cap C, B \cup D).$$

# Union

Union of generalized sets $X$, $Y$:

$$\nu_{X \cup_g Y}(z) = \max(\nu_X(z), \nu_Y(z)), \quad z \in U.$$

A table:

| $\nu_X(z)$ \ $\nu_Y(z)$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $-1$ | $-1$ | $0$ | $1$ |
| $0$ | $0$ | $0$ | $1$ |
| $1$ | $1$ | $1$ | $1$ |

# Union

| $\nu_Y(z)$ / $\nu_X(z)$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $-1$ | $-1$ | $0$ | $1$ |
| $0$ | $0$ | $0$ | $1$ |
| $1$ | $1$ | $1$ | $1$ |

Given $X = (A, B)$, $Y = (C, D)$, where $A \cap B = \varnothing$, $C \cap D = \varnothing$, then:

$$(A, B) \cup_g (C, D) = (A \cup C, B \cap D).$$

# Three-valued logic

Such pairs of sets are studied in the context of three-valued logic and are called inexact classes or orthopairs.

Grzegorz Malinowski, *Kleene logic and inference*, Bulletin of the Section of Logic, University of Łódź, 43 (2014), 43–52.

Davide Ciucci, Didier Dubois, Jonathan Lawry, *Borderline vs. unknown: comparing three-valued representations of imperfect information*, International Journal of Approximate Reasoning 55 (2014), 1866–1889.

# Easy way – $\mathbb{N} \times \mathbb{N}$

Recall the inclusion:

$$(A, B) \subset_g (C, D) \Leftrightarrow A \subset C \wedge D \subset B.$$

If we define maps between finite generalized sets $X = (A, B)$, $Y = (C, D)$, where $A \cap B = \varnothing$, $C \cap D = \varnothing$, as pairs of maps

$$A \to C, \quad D \to B,$$

then the equivalence classes will be $\mathbb{N} \times \mathbb{N}$.

# Instantiation of Schanuel's question

Is it possible to define in some natural way maps between finite generalized sets in order to obtain a category extending the category of finite sets, where $(A, B)$ are $(C, D)$ isomorphic if and only if

$$|A| - |B| = |C| - |D|?$$

# Attention!

If we define maps between finite generalized sets $X = (A, B)$, $Y = (C, D)$, where $A \cap B = \varnothing$, $C \cap D = \varnothing$, as

$$A \sqcup C \to B \sqcup D,$$

then there arize problems with compositions.

# A hint?

A natural candidate for a direct product of generalized sets $(A, B)$ and $(C, D)$ is

$$(A \times C \sqcup B \times D, A \times D \sqcup B \times C).$$

A natural candidate for a direct sum is

$$(A \sqcup C, B \sqcup D).$$

# Two questions

**1.** Do you know any similar construction in some category, where two pairs of objects $(A, B)$ and $(C, D)$ are isomorphic (as objects of the new category) if and only if $A \oplus D$ and $B \oplus C$ are isomorphic in the old category?

**2.** Do you know any construction in some category, where a morphism between pairs of objects $(A, B)$ and $(C, D)$ is defined as a morphism between $A \oplus D$ and $B \oplus C$?

# Thank you very much

# for your attention!!!