Generalizing Principal Bundles

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Outline

References

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Moerdijk's Definition

Let $\mathrm{Sh}(X)$ denote the category of sheaves on a topological space X.

Definition

A $\mathscr{C}$-principal bundle is a functor $Q\colon \mathscr{C} \to \mathrm{Sh}(X)$ such that for each point $x \in X$

1. there is a $c \in \mathscr{C}_0$ for which the stalk $Q(c)_x \neq \emptyset$;
2. for any $q \in Q(c)_x$ and $r \in Q(d)_x$ there is a $b \in \mathscr{C}_0$, a span $c \xleftarrow{f} b \xrightarrow{g} d$ in $\mathscr{C}$ and a $z \in Q(b)_x$ such that $Q(f)(z) = q$ and $Q(g)(z) = r$; and
3. for parallel arrows $f, g\colon c \rightrightarrows d$ and $q \in Q(c)_x$ for which $Q(f)(q) = Q(g)(q)$, there is an arrow $e\colon b \to c$ with $fe = ge$ and a $z \in Q(b)_x$ such that $Q(e)(z) = q$.

Condition 2. is transitivity and 3. is freeness.

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Guiding Question

If Q is instead a pseudo-functor valued in a 2-category, what is a principal bundle?

Case of interest: indexed categories $[\mathscr{X}^{op}, \mathfrak{Cat}]$ on a small category $\mathscr{X}$.

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Theorem

There is an isomorphism

$$\mathbf{Prin}(\mathscr{C}) \cong \mathbf{Geom}(\mathrm{Sh}(X), [\mathscr{C}^{op}, \mathbf{Set}]).$$

Any functor $Q\colon \mathscr{C} \to \mathrm{Sh}(X)$ admits a tensor product $- \otimes_{\mathscr{C}} Q$ extension, which preserves finite limits if, and only if, Q is a principal bundle.

This is proved in [Moe95].

In this sense, the presheaf topos $[\mathscr{C}^{op}, \mathbf{Set}]$ classifies $\mathscr{C}$-principal bundles.

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Tensor Product of Presheaves

Any functor $Q\colon \mathscr{C} \to \mathscr{E}$ on small $\mathscr{C}$ to a cocomplete topos $\mathscr{S}$ admits a tensor product extension along the Yoneda embedding

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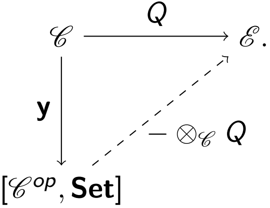

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The functor $-\otimes_{\mathscr{C}} Q$ is one half of a tensor-hom adjunction

$$\mathscr{E}(P \otimes_{\mathscr{C}} Q, X) \cong [\mathscr{C}^{op}, \mathbf{Set}](P, \mathscr{E}(Q, X)).$$

Theorem

The tensor-functor $-\otimes_{\mathscr{C}} Q$ arising from $Q\colon \mathscr{C} \to \mathscr{E}$ preserves finite limits if, and only if, Q is filtering.

Such a functor Q is "flat." In the case that $\mathscr{E}$ is **Set** the functor Q is flat if and only if its category of elements $\int_{\mathscr{C}} Q$ is filtered.

Theorem

There is an equivalence

$$\mathbf{Flat}(\mathscr{C}, \mathscr{E}) \simeq \mathbf{Geom}(\mathscr{E}, [\mathscr{C}^{op}, \mathbf{Set}]).$$

This is Theorem VII.5.2 of [MLM92].

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Outline of Our Approach

- Start with a bimodule $Q\colon \mathscr{X}^{op} \times \mathscr{C} \to \mathfrak{Cat}$, pseudo-functorial in each argument, satisfying a strict interchange law. This yields a transpose
$$\hat{Q}\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}].$$
- Abstract conditions 2. and 3. of Moerdijk's definition to the case of $\hat{Q}$ by weakening the equalities to isomorphisms.
- Construct an extension
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- Investigate the way in which a tensor-hom adjunction, a limit-preserving extension along the Yoneda, and a classifying category are recovered.
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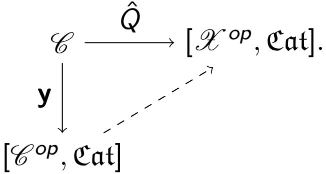

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Main Construction

- Start with pseudo-functors $Q \colon \mathscr{C} \to \mathfrak{Cat}$ and $P \colon \mathscr{C}^{op} \to \mathfrak{Cat}$.
- Set $\Delta(P, Q)$ to be the category with objects triples

$$(c, p, q) \qquad p \in P(c)_0,\ q \in Q(c)_0$$

and arrows $(c, p, q) \to (d, r, s)$ the triples (f, u, v) with

$$f \colon c \to d \qquad u \colon p \to Pf(r) \qquad v \colon Qf(q) \to s.$$

- Take $P \star Q$ to denote the category of fractions

$$P \star Q := \Delta(P, Q)[\Sigma^{-1}]$$

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- For any pseudo-functor $P\colon \mathscr{C}^{op} \to \mathfrak{Cat}$, define a pseudo-functor $\mathscr{X}^{op} \to \mathfrak{Cat}$ by assigning

$$x \mapsto P \star Q(x, -)$$

  on objects with the induced assignments on arrows and identity cells.
- This yields a 2-functor

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$$- \star \hat{Q}\colon [\mathscr{C}^{op}, \mathfrak{Cat}] \longrightarrow [\mathscr{X}^{op}, \mathfrak{Cat}].$$

Tensor-Hom Adjunction

In general, $- \star \hat{Q}$ is a left 2-adjoint. The right adjoint is

$$[\mathscr{X}^{op}, \mathfrak{Cat}](\hat{Q}, -) \colon [\mathscr{X}^{op}, \mathfrak{Cat}] \longrightarrow [\mathscr{C}^{op}, \mathfrak{Cat}].$$

Theorem

For any bimodule Q there is an isomorphism of categories

$$[\mathscr{X}^{op}, \mathfrak{Cat}](P \star \hat{Q}, F) \cong [\mathscr{C}^{op}, \mathfrak{Cat}](P, [\mathscr{X}^{op}, \mathfrak{Cat}](\hat{Q}, F)).$$

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Further Properties

- For any $c \in \mathscr{C}_0$, there is a pseudo-natural equivalence

$$\hat{Q}c \simeq \mathbf{y}c \star \hat{Q}$$

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- In the case $\mathscr{X} = \mathbf{1}$, the construction $P \star Q$ admits a right calculus of fractions if Q is a principal bundle. (Definition to come.)

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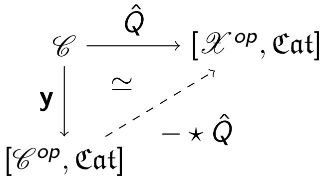

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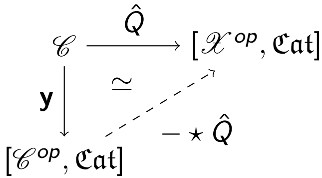

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Pseudo-Coequalizers

The tensor product $P \otimes_{\mathscr{C}} Q$ of ordinary presheaves fits into a coequalizer diagram of the form

$$P \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} Q \underset{\alpha' \times 1}{\overset{1 \times \alpha}{\rightrightarrows}} P \times_{\mathscr{C}_0} Q \dashrightarrow P \otimes_{\mathscr{C}} Q.$$

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For pseudo-functors P and Q, the category of fractions $P \star Q$ fits into a pseudo-coequalizer diagram

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A bimodule $Q\colon \mathscr{X}^{op} \times \mathscr{C} \to \mathfrak{Cat}$ is a $\mathscr{C}$-principal bundle over $\mathscr{X}$ provided that for each $x \in \mathscr{X}_0$, each $Q(x,c)$ is in $\mathfrak{Gpd}$ and

1. there is $c \in \mathscr{C}_0$ such that $Q(x,c)$ is nonempty;
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- The definition is essentially that each $Q(x, c)$ is a groupoid and for each $x \in \mathscr{X}_0$, the Grothendieck completion

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  is filtered.

- When $\mathscr{X}$ is just $\mathbf{1}$, there is just the pseudo-functor $Q\colon \mathscr{C} \to \mathfrak{Cat}$, which is a $\mathscr{C}$-principal bundle if and only if Q is fibred in $\mathfrak{Gpd}$ and the completion $\int_{\mathscr{C}} Q$ is filtered.
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- Weighted pseudo-limits can be constructed from finite products, pseudo-equalizers, and cotensors with **2**.
- For F valued in $[\mathscr{X}^{op}, \mathfrak{Cat}]$, there is an induced canonical functor from the image of a limit to the limit of the images. For example, binary products

- Say that a pseudo-functor (valued in $[\mathscr{X}^{op}, \mathfrak{Cat}]$) "essentially preserves" a type of finite pseudo-limit if (the components of) the corresponding canonical functors are essentially surjective.

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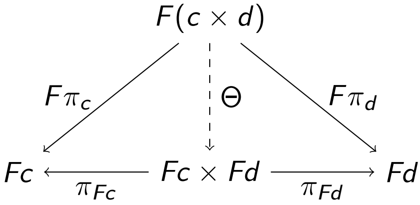

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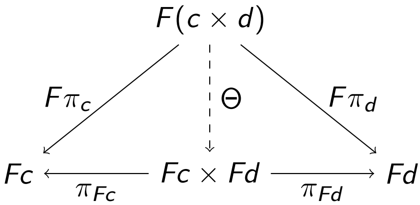

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Remarks on the Proof

- Can reduce to the case where $\mathscr{X}$ is $\mathbf{1}$.
- The proof follows a pattern: fibred in $\mathfrak{Gpd}$ corresponds to essential preservation of cotensors with $\mathbf{2}$; nontriviality corresponds to preservation of $\mathbf{1}$; transitivity to essential preservation of binary products; and freeness to preservation of equalizers.
- Proof of sufficiency uses only representables, more-or-less replicating the proof that flat implies filtered in VII.6.3 of [MLM92].
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Indexed Categories Classify Generalized Principal Bundles

- Let $\mathfrak{Prin}(\mathscr{C})$ denote the 2-category of $\mathscr{C}$-principal bundles.
- Let $\mathfrak{Hom}(\mathfrak{Cat}, [\mathscr{C}^{op}, \mathfrak{Cat}])$ denote the 2-category of 2-adjunctions

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whose left adjoints essentially preserve finite limits.

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