Integration in Tangent Categories

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Work with Robin Cockett and Geoff Cruttwell

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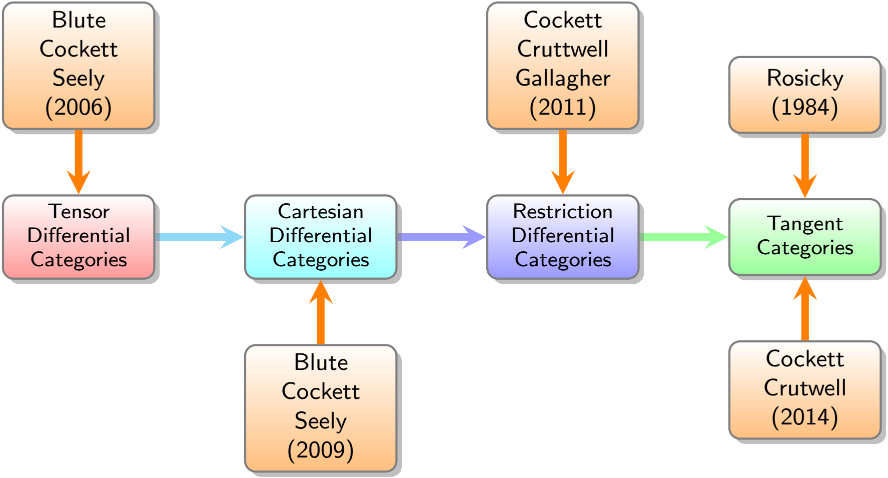
A Story of Differential Categories



Blute Cockett Seely (2006)

Tensor Differential Categories

Cartesian Differential Categories

Blute Cockett Seely (2009)

Cockett Cruttwell Gallagher (2011)

Restriction Differential Categories

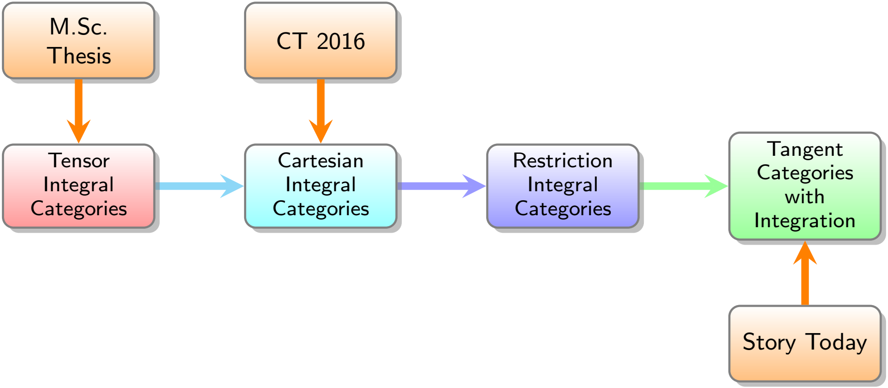
Rosicky (1984)

Tangent Categories

Cockett Crutwell (2014)

A Story of Integral Categories

We are trying to get the dual story of integration, in the context of antiderivatives and which give **fundamental theorems of calculus**:



M.Sc. Thesis

CT 2016

Tensor Integral Categories

Cartesian Integral Categories

Restriction Integral Categories

Tangent Categories with Integration

Story Today

What are we looking for?

	Differential	Integration	2nd Fund. Thm.
Tensor			
Cartesian			
Tangent			

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[1] Composition is written diagrammatically

What are we looking for?

	Differential	Integration	2nd Fund. Thm.
Tensor	Deriving Transformation $\mathsf{d} : !A \otimes A \to !A$	Integral Transformation $\mathsf{s} : !A \to !A \otimes A$	$\mathsf{sd} + !(0) = 1$[1]
Cartesian			
Tangent			

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[1] Composition is written diagrammatically

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Cartesian	Differential Combinator $\dfrac{f : A \to B}{\mathsf{D}[f] : A \times \underset{\text{Linear}}{A} \to B}$	Integral Combinator $\dfrac{g : A \times \overset{\text{Linear}}{A} \to B}{\mathsf{S}[g] : A \to B}$	$\mathsf{S}[\mathsf{D}[f]] + 0f = f$
Tangent			

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[1] Composition is written diagrammatically

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Tangent	Tangent Functor $\dfrac{f : M \to N}{\mathsf{T}(f) : \mathsf{T}(M) \to \mathsf{T}(N)}$	?	?

---

[1] Composition is written diagrammatically

Tangent Categories

A **tangent category** is a category $\mathbb{X}$ equipped with:

- A functor $\mathsf{T} : \mathbb{X} \to \mathbb{X}$ called the **tangent functor**;
- A natural transformation $\mathsf{p} : \mathsf{T}(M) \to M$;
- A natural transformation $\ell : \mathsf{T}(M) \to \mathsf{T}^2(M)$ called the **vertical lift**;
- A natural transformation $\mathsf{c} : \mathsf{T}^2(M) \to \mathsf{T}^2(M)$ called the **canonical flip**.

such that:

- $\mathsf{p} : \mathsf{T}(M) \to M$ is a commutative monoid in $\mathbb{X}/M$ where the addition $+ : \mathsf{T}_2(M) \to \mathsf{T}(M)$ (where $\mathsf{T}_2(M)$ is the pullback of p along itself) and the unit $0 : M \to \mathsf{T}(M)$ are natural transformations;
- Various other properties of and coherences between T, p, ℓ, c, $+$, 0.

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- Various other properties of and coherences between T, p, ℓ, c, $+$, 0.

A **cartesian tangent category** is a tangent category with finite products which are preserved by the tangent funtor:

$$\mathsf{T}(A \times B) \cong \mathsf{T}(A) \times \mathsf{T}(B)$$

Cartesian tangent categories have a natural strength map $\theta : C \times \mathsf{T}(A) \to \mathsf{T}(C \times A)$ defined as:

$$C \times \mathsf{T}(A) \xrightarrow{0 \times 1} \mathsf{T}(C) \times \mathsf{T}(A) \cong \mathsf{T}(C \times A)$$

Examples of Tangent Categories

Example

- Every category (with finite products) is a (cartesian) tangent category where the tangent functor is the identity functor;
- The category of finite-dimensional smooth manifolds is a cartesian tangent category where for a manifold M, $\mathrm{T}(M)$ is its tangent bundle;
- A model of SDG with an object of infinitesimals D, the category of microlinear objects is a cartesian tangent category with $\mathrm{T}(M) = M^D$;
- Many other examples given by Geoff, Robin, Jonathan and Ben.

Differential Objects

In a cartesian tangent category, a **differential object** is a commutative monoid $(A, \sigma : A \times A \to A, \mathsf{z} : 1 \to A)$ equipped with a map:

$$\hat{\mathsf{p}} : \mathsf{T}(A) \to A$$

such that:

- $A \xleftarrow{\mathsf{p}} \mathsf{T}(A) \xrightarrow{\hat{\mathsf{p}}} A$ is a product diagram, so in particular $\mathsf{T}(A) \cong A \times A$;
- Various coherences between σ, z and $\hat{\mathsf{p}}$ with the tangent structure.

Example

- Differential objects for the category of smooth manifolds are the cartesian spaces $\mathbb{R}^n$:

$$\mathsf{T}(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^n$$

Linear in Context Bundle Morphisms

A tangent bundle morphism in context C, i.e, a commutative square:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \arrow[r, "f"] \arrow[d, "1\times \mathsf{p}"'] & \mathsf{T}(N) \arrow[d, "\mathsf{p}"] \\
C \times M \arrow[r, "g"'] & N
\end{tikzcd}$$

is **linear in context** C if the following diagram commutes:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \arrow[rr, "f"] \arrow[d, "1\times \ell"'] & & \mathsf{T}(N) \arrow[d, "\ell"] \\
C \times \mathsf{T}^2(M) \arrow[r, "\theta"'] & \mathsf{T}(C \times \mathsf{T}(M)) \arrow[r, "\mathsf{T}(f)"'] & \mathsf{T}^2(N)
\end{tikzcd}$$

Properties of Linear Bundle Morphism

- For every $f : C \times M \to N$, the following is a linear bundle morphism:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \arrow[r, "\theta"] \arrow[d, "\mathsf{p}"'] & \mathsf{T}(C \times M) \arrow[r, "\mathsf{T}(f)"] & \mathsf{T}(N) \arrow[d, "\mathsf{p}"] \\
\mathsf{T}(M) \arrow[rr, "f"'] & & \mathsf{T}(N)
\end{tikzcd}$$

- Composition of linear bundle morphisms is a linear bundle morphism:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \arrow[r, "\langle \pi_0, f \rangle"] \arrow[d, "\mathsf{p}"'] & C \times \mathsf{T}(N) \arrow[r, "k"] \arrow[d, "\mathsf{p}"] & \mathsf{T}(Q) \arrow[d, "\mathsf{p}"] \\
C \times M \arrow[r, "\langle \pi_0, g \rangle"'] & C \times \mathsf{T}(N) \arrow[r, "h"'] & Q
\end{tikzcd}$$

- Sum of linear bundle morphisms (over the same base) is a linear bundle morphism:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \arrow[r, "\langle f, h \rangle"] \arrow[d, "\mathsf{p}"'] & \mathsf{T}_2(N) \arrow[r, "+"] & \mathsf{T}(N) \arrow[d, "\mathsf{p}"] \\
C \times M \arrow[rr, "g"'] & & \mathsf{T}(N)
\end{tikzcd}$$

Bilinear in Context Bundle Morphisms

A bundle morphism:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \times \mathsf{T}(M') \arrow[r, "f"] \arrow[d, "1\times\mathsf{p}\times\mathsf{p}"'] & \mathsf{T}(N) \arrow[d, "\mathsf{p}"] \\
C \times M \times M' \arrow[r, "g"'] & N
\end{tikzcd}$$

is **bilinear in context** C if f is both linear in context $C \times \mathsf{T}(M)$ and in context $C \times \mathsf{T}(M')$, that is, the following diagrams commute:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \times \mathsf{T}(M') \arrow[rr, "f"] \arrow[d, "1\times\ell\times 1"'] & & \mathsf{T}(N) \arrow[d, "\ell"] \\
C \times \mathsf{T}^2(M) \times \mathsf{T}(M') \arrow[r, "\theta"'] & \mathsf{T}(C \times \mathsf{T}(M) \times \mathsf{T}(M')) \arrow[r, "\mathsf{T}(f)"'] & \mathsf{T}^2(N)
\end{tikzcd}$$

$$\begin{tikzcd}
C \times \mathsf{T}(M) \times \mathsf{T}(M') \arrow[rr, "f"] \arrow[d, "1\times 1\times\ell"'] & & \mathsf{T}(N) \arrow[d, "\ell"] \\
C \times \mathsf{T}(M) \times \mathsf{T}^2(M') \arrow[r, "\theta"'] & \mathsf{T}(C \times \mathsf{T}(M) \times \mathsf{T}(M')) \arrow[r, "\mathsf{T}(f)"'] & \mathsf{T}^2(N)
\end{tikzcd}$$

Integration for Cartesian Tangent Categories

A cartesian tangent category has **integration** for a class of objects $\mathcal{I}$ which is:

- Closed under the tangent functor, i.e, if $M \in \mathcal{I}$ then $\mathsf{T}(M) \in \mathcal{I}$;
- Closed under product, i.e, if $M, N \in \mathcal{I}$ then $M \times N \in \mathcal{I}$;
- Contains the differential objects.

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if for each linear in context bundle morphism with in context domain in $\mathcal{I}$, i.e, $M \in \mathcal{I}$:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \arrow[r, "f"] \arrow[d, "\mathsf{p}"'] & \mathsf{T}(N) \arrow[d, "\mathsf{p}"] \\
C \times M \arrow[r, "g"'] & N
\end{tikzcd}$$

there exists a map which makes the following (lower) triangle commute:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \arrow[r, "f"] \arrow[d, "1 \times \mathsf{p}"'] & \mathsf{T}(N) \arrow[d, "\mathsf{p}"] \\
C \times M \arrow[r, "g"'] \arrow[ur, "\mathsf{S}_M[f,g]"] & N
\end{tikzcd}$$

and satisfies the following axioms...

Axiom: Preserves Linearity

If the following bundle morphism is bilinear in context C:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \times \mathsf{T}(M') \arrow[r, "f"] \arrow[d, "1\times \mathsf{p} \times \mathsf{p}"'] & \mathsf{T}(N) \arrow[d, "\mathsf{p}"] \\
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\end{tikzcd}$$

Then the integral of:

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C \times \mathsf{T}(M) \times \mathsf{T}(M') \arrow[r, "f"] \arrow[d, "1\times \mathsf{p} \times 1"'] & \mathsf{T}(N) \arrow[dd, "\mathsf{p}"] \\
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C \times M \times \mathsf{T}(M') \arrow[ur] \arrow[d, "1\times 1\times\mathsf{p}"'] & \\
C \times M \times M' \arrow[r, "g"'] & N
\end{tikzcd}$$

is a linear in context $C \times M$ bundle morphism.

Axioms: Preserves Additivity and the Linear Scaling Rule

- The integral preserves the additive structure:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \arrow[r, "\langle f, h\rangle"] \arrow[d, "\mathsf{p}"'] & \mathsf{T}_2(N) \arrow[r, "+"] & \mathsf{T}(N) \arrow[d, "\mathsf{p}"] \\
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C \times M \arrow[rru] \arrow[rr, "g"'] & & \mathsf{T}(N)
\end{tikzcd}$$

$$\mathsf{S}_M[\langle f, h \rangle +, g] = \langle \mathsf{S}_M[f, g], \mathsf{S}_M[h, g] \rangle +$$

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- The integral preserves composition on the right by linear bundle morphisms:

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C \times M \arrow[ru] \arrow[rru] \arrow[r, "g"'] & N \arrow[r, "h"'] & Q
\end{tikzcd}$$

$$\mathsf{S}_M[f, g]k = \mathsf{S}_M[fk, gh]$$

Axiom: Rota-Baxter Rule

In classical calculus, the Rota-Baxter rule [2] is integration by parts without derivatives:

$$(\int f \; \mathrm{d}x) \cdot (\int g \; \mathrm{d}x) = \int (\int f \; \mathrm{d}u) \cdot g \; \mathrm{d}x + \int f \cdot (\int g \; \mathrm{d}u) \; \mathrm{d}x$$

Expressing this in tangent categories is a little tricky and messy...

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[2] Thanks to Rick Blute for introducing this idea to me

Axiom: Rota-Baxter Rule

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$$\begin{tikzcd}
C \times \mathsf{T}(M) \times \mathsf{T}(M') \arrow[r, "f"] \arrow[d, "1\times \mathsf{p}\times \mathsf{p}"'] & \mathsf{T}(N) \arrow[d, "\mathsf{p}"] \\
C \times M \times M' \arrow[r, "g"'] & N
\end{tikzcd}$$

Then the double integral:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \times \mathsf{T}(M') \arrow[r, "f"] \arrow[d, "1\times \mathsf{p}\times 1"'] & \mathsf{T}(N) \arrow[dd, "\mathsf{p}"] \\
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C \times M \times \mathrm{T}(M') \arrow[ur] \arrow[d, "1\times 1\times \mathrm{p}"'] & \\
C \times M \times M' \arrow[r, "g"'] \arrow[uur] & N
\end{tikzcd}$$

$$\mathrm{S}_{M'}[\mathrm{S}_M[f, (1 \times 1 \times \mathrm{p})g], g] = \ldots$$

is equal to the sum of the following double integrals:

Axiom: Rota-Baxter Rule

$$\begin{tikzcd}
C \times \mathsf{T}(M \times M') \arrow[r, "\cong"] \arrow[ddr, "1\times\mathsf{p}"'] & C \times \mathsf{T}(M) \times \mathsf{T}(M') \arrow[r, "f"] \arrow[d, "1\times\mathsf{p}\times 1"'] & \mathsf{T}(N) \arrow[dd, "\mathsf{p}"] \\
 & C \times M \times \mathsf{T}(M') \arrow[d, "1\times 1\times\mathsf{p}"'] & \\
 & C \times M \times M' \arrow[r, "g"'] & N
\end{tikzcd}$$

$$\begin{tikzcd}
C \times \mathsf{T}(M \times M') \arrow[r, "\cong"] \arrow[ddr, "1\times\mathsf{p}"'] & C \times \mathsf{T}(M) \times \mathsf{T}(M') \arrow[r, "f"] \arrow[d, "1\times 1\times\mathsf{p}"'] & \mathsf{T}(N) \arrow[dd, "\mathsf{p}"] \\
 & C \times \mathsf{T}(M) \times M' \arrow[d, "1\times\mathsf{p}\times 1"'] & \\
 & C \times M \times M' \arrow[r, "g"'] & N
\end{tikzcd}$$

Axiom: Rota-Baxter Rule

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& C \times M \times \mathsf{T}(M') \arrow[ur] \arrow[d, "1\times1\times\mathsf{p}"'] & \\
& C \times M \times M' \arrow[r, "g"'] & N
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& C \times \mathsf{T}(M) \times M' \arrow[ur] \arrow[d, "1\times\mathsf{p}\times1"'] & \\
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Axiom: Rota-Baxter Rule

$$\begin{tikzcd}
C \times \mathrm{T}(M \times M') \arrow[r, "\cong"] \arrow[ddr, swap, "1 \times \mathrm{p}"] & C \times \mathrm{T}(M) \times \mathrm{T}(M') \arrow[r, gray, "f"] \arrow[d, swap, "1 \times \mathrm{p} \times 1"] & \mathrm{T}(N) \arrow[dd, "\mathrm{p}"] \\
 & C \times M \times \mathrm{T}(M') \arrow[ur, red] \arrow[d, gray, swap, "1 \times 1 \times \mathrm{p}"] & \\
 & C \times M \times M' \arrow[r, swap, "g"] & N
\end{tikzcd}$$

$$\begin{tikzcd}
C \times \mathrm{T}(M \times M') \arrow[r, "\cong"] \arrow[ddr, swap, "1 \times \mathrm{p}"] & C \times \mathrm{T}(M) \times \mathrm{T}(M') \arrow[r, gray, "f"] \arrow[d, swap, "1 \times 1 \times \mathrm{p}"] & \mathrm{T}(N) \arrow[dd, "\mathrm{p}"] \\
 & C \times \mathrm{T}(M) \times M' \arrow[ur, red] \arrow[d, gray, swap, "1 \times \mathrm{p} \times 1"] & \\
 & C \times M \times M' \arrow[r, swap, "g"] & N
\end{tikzcd}$$

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 & C \times M \times M' \arrow[r, "g"'] \arrow[uur] & N
\end{tikzcd}$$

$$\begin{tikzcd}
C \times \mathrm{T}(M \times M') \arrow[r, "\cong"] \arrow[ddr, "1\times\mathrm{p}"'] & C \times \mathrm{T}(M) \times \mathrm{T}(M') \arrow[r, "f"] \arrow[d, "1\times 1\times\mathrm{p}"'] & \mathrm{T}(N) \arrow[dd, "\mathrm{p}"] \\
 & C \times \mathrm{T}(M) \times M' \arrow[d, "1\times\mathrm{p}\times 1"'] \arrow[ur] & \\
 & C \times M \times M' \arrow[r, "g"'] \arrow[uur] & N
\end{tikzcd}$$

Axiom: Rota-Baxter Rule

$$\mathsf{S}_{M'}[\mathsf{S}_M[f,(1\times 1\times \mathsf{p})g],g] =$$
$$\langle \mathsf{S}_{M\times M'}[(1\times 1\times \mathsf{p})\mathsf{S}_M[f,(1\times 1\times \mathsf{p})g],g], \mathsf{S}_{M\times M'}[(1\times \mathsf{p}\times 1)\mathsf{S}_{M'}[f,(1\times \mathsf{p}\times 1)g],g]\rangle +$$

Remark

Hiding in the Rota-Baxter Rule is Fubini's Theorem:

$$\mathsf{S}_{M'}[\mathsf{S}_M[f,(1\times 1\times \mathsf{p})g],g] = \mathsf{S}_M[\mathsf{S}_{M'}[f,(1\times \mathsf{p}\times 1)g],g]$$

Axiom: Interchange Rule

If the following linear in context C bundle morphism:

$$\begin{tikzcd}
C \times \mathsf{T}^2(M) \arrow[rr, "f"] \arrow[d, "1\times \mathsf{p}"'] & & \mathsf{T}(N) \arrow[d, "\mathsf{p}"] \\
C \times \mathsf{T}(M) \arrow[r, "1\times \mathsf{p}"'] & C \times M \arrow[r, "g"'] & N
\end{tikzcd}$$

and the following diagram commutes:

$$\begin{tikzcd}
C \times \mathsf{T}^2(M) \arrow[rr, "f"] \arrow[d, "1\times \mathsf{T}(\ell)"'] & & \mathsf{T}(N) \arrow[dd, "\ell"] \\
C \times \mathsf{T}^3(M) \arrow[d, "1\times \mathsf{c}"'] & & \\
C \times \mathsf{T}^3(M) \arrow[r, "\theta"'] & \mathsf{T}(C \times \mathsf{T}^2(M)) \arrow[r, "\mathsf{T}(f)"'] & \mathsf{T}^2(N)
\end{tikzcd}$$

Then...

Axiom: Interchange Rule

Then the integral:

$$\begin{tikzcd}
C \times \mathrm{T}^2(M) \arrow[rr, "f"] \arrow[d, "1\times \mathrm{p}"'] & & \mathrm{T}(N) \arrow[dd, "\mathrm{p}"] \\
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\end{tikzcd}$$

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is a linear in context C bundle morphism. And the double integral satisfies:

$$\mathsf{S}_M[\mathsf{S}_{\mathsf{T}(M)}[f,(1\times\mathsf{p})g],g] = \mathsf{S}_M[\mathsf{S}_{\mathsf{T}(M)}[(1\times\mathsf{c})f,(1\times\mathsf{p})g],g]$$

Axioms for Differential Objects: Constant Rule and Linear Substitution Rule

- If A is a differential object then:

$$\begin{tikzcd}
\mathsf{T}(A) \arrow[rr, equal] \arrow[dd, "\mathsf{p}"'] & & \mathsf{T}(A) \arrow[dd, "\mathsf{p}"] \\
& A \times A \arrow[ur, "\cong"] & \\
A \arrow[rr, equal] \arrow[ur, "\Delta"] & & A
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- If a map between differential objects $h : A \to B$ satisfies that $\mathsf{T}(h) = \langle \mathsf{p}h, \hat{\mathsf{p}}h \rangle$, then:

$$\begin{tikzcd}
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A \arrow[r, "h"'] \arrow[urr] & B \arrow[r, "g"'] \arrow[ur] & N
\end{tikzcd}$$

$$\mathsf{S}_A[\mathsf{T}(h)f, hg] = h\mathsf{S}_B[f, g]$$

Axiom: Poincaré Condition

Let A be a differential object. If a linear in context bundle morphism:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \arrow[r, "f"] \arrow[d, "1\times \mathsf{p}"'] & \mathsf{T}(A) \arrow[d, "\mathsf{p}"] \\
C \times M \arrow[r, "g"'] \arrow[ur, "\mathsf{S}_M[f{,}g]"] & A
\end{tikzcd}$$

satisfies $\theta\mathsf{T}(f) = (1 \times \mathsf{c})\theta\mathsf{T}(f)$ then the following diagram commutes:

$$\begin{tikzcd}
C \times \mathsf{T}(M) \arrow[r, "\theta"] \arrow[drr, "f"'] & \mathsf{T}(C \times M) \arrow[r, "\mathsf{T}(\mathsf{S}_M[f{,}g])"] & \mathsf{T}^2(A) \arrow[d, "\mathsf{T}(\hat{\mathsf{p}})"] \\
& & \mathsf{T}(A)
\end{tikzcd}$$

Axiom: Fundamental Theorem of Calculus

Let A and B be differential objects. Then for every map $f : C \times A \to B$,

$$\begin{tikzcd}
C \times \mathsf{T}(A) \arrow[r, "\theta"] \arrow[d, "1\times \mathsf{p}"'] & \mathsf{T}(C \times M) \arrow[r, "\mathsf{T}(f)"] & \mathsf{T}(B) \arrow[d, "\mathsf{p}"] \\
C \times A \arrow[rr, "f"'] \arrow[urr, red, "\mathsf{S}_A[\theta \mathsf{T}(f), f]"] & & B
\end{tikzcd}$$

the following equality holds:

$$\langle \mathsf{S}_A[\theta \mathsf{T}(f), f]\hat{\mathsf{p}}, (1 \times \mathsf{z}_A)f \rangle \sigma_B = f$$

Example [2]: Integration on Star-Shaped Open Subsets

Define the cartesian tangent category of real open subsets as follows:

- The objects are open subsets of $\mathbb{R}^n$: $(U \subseteq \mathbb{R}^n)$;
- The maps are smooth functions between open subsets;
- Identity, composition and products are standard;
- The tangent functor on objects given:

  $$\mathsf{T}(U \subseteq \mathbb{R}^n) = (U \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n)$$

  While on maps gives:

  $$\begin{aligned} \mathsf{T}(f) : \mathsf{T}(U \subseteq \mathbb{R}^n) &\to \mathsf{T}(V \subseteq \mathbb{R}^m) \\ (u, v) &\mapsto (f(u), \mathsf{D}[f](u, v)) \end{aligned}$$

  where $\mathsf{D}[f](u, v)$ is the directional derivative of f at point u in direction v.
- The differential objects are the cartesian spaces $(\mathbb{R}^n \subseteq \mathbb{R}^n)$

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Example

Define the class of objects $\mathcal{I}$ of star-shaped at 0 open subsets:

$$U \in \mathcal{I} \Leftrightarrow \forall u \in U,\ \forall t \in [0, 1].\ tu \in U$$

---

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Example: Integration on Star-Shaped Open Subsets

A bundle morphism:

$$\begin{tikzcd}
C \times \mathsf{T}(U \subseteq \mathbb{R}^n) \arrow[r, "\langle \pi_0 g, f \rangle"] \arrow[d, "1 \times \mathsf{p}"'] & \mathsf{T}(V \subseteq \mathbb{R}^m) \arrow[d, "\mathsf{p}"] \\
C \times (U \subseteq \mathbb{R}^n) \arrow[r, "g"'] & (V \subseteq \mathbb{R}^m)
\end{tikzcd}$$

is linear in context if there exists a smooth map $F : C \times U \to \mathbb{R}^n$ such that:

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Example

Suppose $U \in \mathcal{I}$, star-shaped at 0, then the integral is defined as:

$$\mathsf{S}_U[\langle \pi_0 g, f \rangle, g] : C \times \mathsf{T}(U \subseteq \mathbb{R}^n) \to \mathsf{T}(V \subseteq \mathbb{R}^m)$$

$$(c, u) \mapsto (g(u), \int_0^1 F(c, tu) \cdot u \, \mathrm{d}t)$$

This is the line integral of F over the straight line path between 0 and u.

This example extends to the category of finite-dimensional smooth manifolds.

Concluding Discussion

- This is **NOT** the end of the story:
    - This captures line integration of 1-forms for a specific path and specific manifolds;
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  - Can we move away from starting our paths at 0?
  - Can we work with any path?
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  - Can we work with any path?
  - Can we expand our class of objects to the entire category?
- A possible place to look for a solution: **Curve Objects**

Thank You

END.

Thank you!