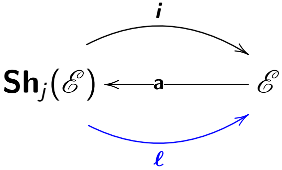
Duality Theorems for Essential Inclusions of Grothendieck Toposes

Guilherme Frederico Lima

CT 2017 : Vancouver

Essential Inclusions

An inclusion of toposes is *essential* if the inverse image functor has a left adjoint:



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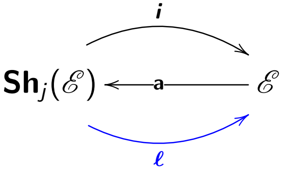

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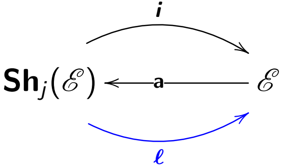

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This says that the category of j-sheaves is equivalent to the category of j-discrete objects:

$$\mathbf{Sh}_j(\mathscr{E}) \simeq \mathbf{D}_j(\mathscr{E})$$

Let $\mathbf{a} \dashv i : \mathbf{Sh}_j(\mathscr{E}) \to \mathscr{E}$ be an inclusion of toposes.

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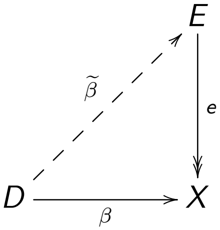

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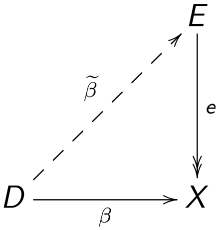
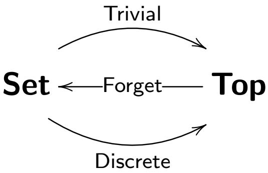


$\mathbf{D}_j(\mathscr{E}) \hookrightarrow \mathscr{E}$ full subcategory of discrete objects.

Motivation:



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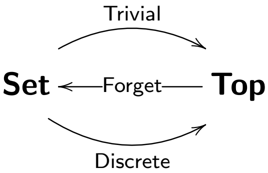

$$\text{Discrete} \dashv \text{Forget} \dashv \text{Trivial}$$

Theorem: (Kelly & Lawvere - 1989):

An inclusion of Grothendieck toposes $\mathbf{Sh}(\mathbb{C}, K) \hookrightarrow \mathbf{Sh}(\mathbb{C}, J)$ is essential iff each $\mathbf{a}(yA) \in \mathbf{Sh}(\mathbb{C}, J)$ has a smallest dense subobject.

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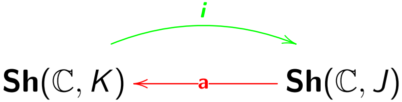
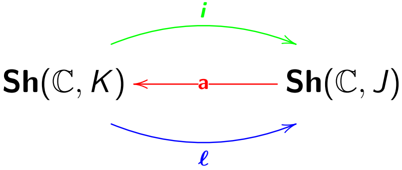
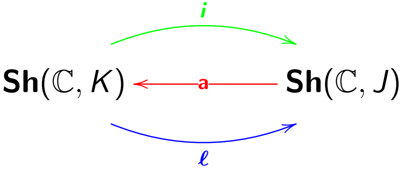

i
Sh(ℂ, K)
a
Sh(ℂ, J)



i
Sh(ℂ, K)
a
Sh(ℂ, J)
ℓ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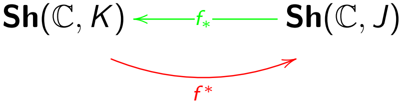
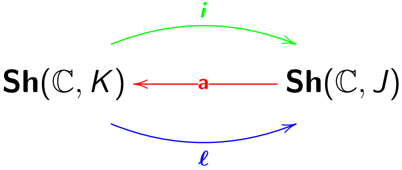

Sh(ℂ, K)
Sh(ℂ, J)
f_*
f^*



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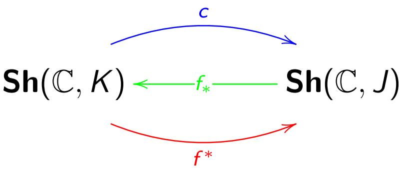
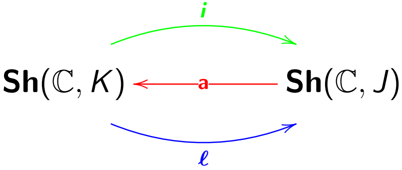
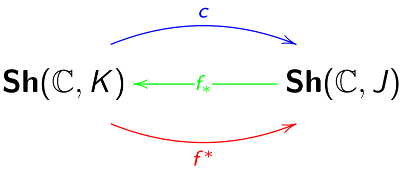


c
Sh(ℂ, K)
f*
Sh(ℂ, J)
f*





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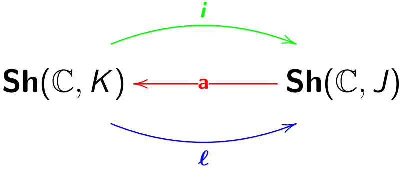

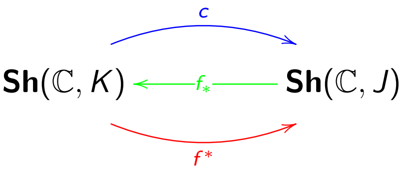

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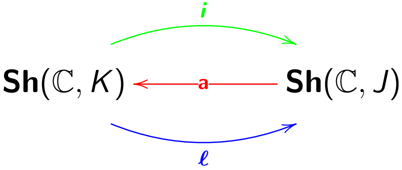

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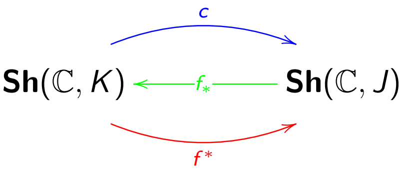

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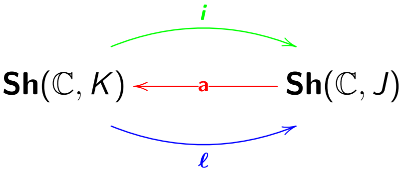

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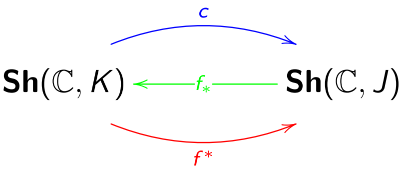

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Local Geometric Morphism

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Theorem: (Kelly & Lawvere - 1989)

Let $\mathbb{C}$ be a small category. There is an order preserving bijection between essential inclusions into $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ and idempotent ideals on $\mathbb{C}$.

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$\mathcal{I} \subseteq \mathbf{Mor}(\mathbb{C})$ is an *ideal* if:

$f \in \mathcal{I} \Rightarrow fg \in \mathcal{I}$ and $f \in \mathcal{I} \Rightarrow hf \in \mathcal{I}$.

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$f \in \mathcal{I} \Rightarrow f = gh$ where $g, h \in \mathcal{I}$.

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But what if $\sigma(A, B) \rightarrowtail \mathbf{a}(yB)(A)$?

$$
\begin{array}{ccc}
\coprod_{B} \mathbb{C}(A,B) \times \mathbb{C}(B,C) & \twoheadrightarrow & \mathbb{C}(A,C) \\
\uparrow & & \uparrow \\
\coprod_{B} \sigma(A,B) \times \sigma(B,C) & \twoheadrightarrow & \sigma(A,C)
\end{array}
$$

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\coprod_B \sigma(-,B)\times\sigma(B,-) \twoheadrightarrow \sigma\otimes\sigma \longrightarrow \sigma
$$

Theorem: (G.F.L. 2016) Let $\mathbf{Sh}(\mathbb{C}, J)$ be a Grothendieck topos. There is an order preserving bijection between essential inclusions into $\mathbf{Sh}(\mathbb{C}, J)$ and subfunctors $\sigma \rightarrowtail \mathbf{a} \circ y : \mathbb{C} \to \mathbf{Sh}(\mathbb{C}, J)$ such that

$$\sigma \otimes \sigma \to \sigma$$

is an epi.

Theorem: (G.F.L. 2016) Let $\mathbb{L}$ be a locale. There is an order preserving bijection between local geometric morphisms *out of* $\mathbf{Sh}(\mathbb{L})$ and finite-limit-preserving subfunctors of the Yoneda embedding $\sigma \rightarrowtail y : \mathbb{L} \to \mathbf{Sh}(\mathbb{L})$ such that

$$\sigma \otimes \sigma \cong \sigma.$$

Theorem: (Johnstone & Moerdijk - 1989)

Let $f : X \to Y$ be a continuous map of sober topological spaces. Then the induced morphism $f : \mathbf{Sh}(X) \to Sh(Y)$ is local if and only if there exists a continuous section $c : Y \to X$ of f with $cf(y) \leq y$ for all $y \in Y$.

—————— $\star$ ——————

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——————— ⋆ ———————

Let $\mathbb{L}$ be a locale. There is an order-preserving bijection between local geometric morphisms out of $\mathbf{Sh}(\mathbb{L})$ and idempotent endomorphisms of locales $\sigma^{-1} : \mathbb{L} \to \mathbb{L}$ which satisfy $\sigma^{-1} \leq \mathrm{id}$.

An inclusion of Grothendieck toposes $\mathbf{Sh}(\mathbb{C}, K) \hookrightarrow \mathbf{Sh}(\mathbb{C}, J)$ is essential iff the closure operation

$$\mathbf{cl} : \mathbf{Sub} \to \mathbf{Sub}$$

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$$\mathbf{int}(X) = \mathbf{im}(\sigma \otimes X \to X)$$

Theorem: (G.F.L. 2016) Let $\mathbf{Sh}(\mathbb{C}, J)$ be a Grothendieck topos. There is an order preserving bijection between essential inclusions into $\mathbf{Sh}(\mathbb{C}, J)$ and endofunctors $\mathbf{int} : \mathbf{Sh}(\mathbb{C}, J) \to \mathbf{Sh}(\mathbb{C}, J)$ such that

$$\mathbf{int} \rightarrowtail \mathrm{id},$$

$$\mathbf{int} \circ \mathbf{int} \cong \mathbf{int},$$

and **int** preserves epis and small coproducts.

Thank you!

References

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