Categorical-algebraic methods in group cohomology

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This is joint work with many people, done over the last 15 years.

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What are the connections between these developments?

Overview, $n = 1$

	Homology $H_2(X)$	Cohomology $H^2(X, (A, \xi))$	
		trivial action ξ	arbitrary action ξ
Gp	$\dfrac{R \wedge [F, F]}{[R, F]}$	$CentrExt^1(X, A)$	$OpExt^1(X, A, \xi)$
abelian categories	0	$Ext^1(X, A)$	
Barr-exact categories		$Tors^1[X, (A, \xi)]$	
semi-abelian categories	$\dfrac{R \wedge [F, F]}{[R, F]}$	$CentrExt^1(X, A)$	$OpExt^1(X, A, \xi)$

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Low-dimensional cohomology of groups, I

An **extension** from A to X is a short exact sequence

$$0 \longrightarrow A \rightarrowtail E \overset{f}{\twoheadrightarrow} X \longrightarrow 0.$$

It is **central** if and only if $[A, E] = 0$: all $eae^{-1}a^{-1}$ vanish, $a \in A$, $e \in E$.

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Theorem [Eckmann 1945-46; Eilenberg & Mac Lane, 1947]

For any abelian group A we have $H^2(X, A) \cong \mathit{CentrExt}^1(X, A)$,
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- By the Short Five Lemma, *equivalence class = isomorphism class*:

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0 \arrow[r] & A \arrow[r, tail] \arrow[d, equal] & E \arrow[r, "f"] \arrow[d, "e"'] & X \arrow[r] \arrow[d, equal] & 0 \\
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The theorem remains true [Gran & VdL, 2008] in any semi-abelian category [Janelidze, Márki & Tholen, 2002] with enough projectives;
centrality may be defined via commutator theory or via categorical Galois theory.

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Theorem (Hopf formula for $H_2(X)$, [Hopf, 1942])

Consider a **projective presentation** $X \cong F/R$ of X: an extension $0 \to R \to F \to X \to 0$ where F is projective. Then the *second integral homology group* $H_2(X)$ is $\frac{R \wedge [F,F]}{[R,F]}$.

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Basic analysis

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- In fact, f is central iff the bottom right square is a pullback.

$$\begin{tikzcd}
Eq(f) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, "f"] \\
F \arrow[r, "f"'] & X
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Eq(f) \arrow[r, "\pi_2"] \arrow[d, "\eta_{Eq(f)}"'] & F \arrow[d, "\eta_F"] \\
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The theorem remains true [Everaert & VdL, 2004] for reflectors of semi-abelian varieties of algebras to their subvarieties: $[X, X]$ is commutator (Gp vs. Ab), Lie bracket ($Lie_{\mathbb{K}}$ vs. $Vect_{\mathbb{K}}$), product XX (Alg_R vs. Mod_R), or …

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$$\begin{tikzcd} X \arrow[r, "s"] \arrow[dr, "1_X"'] & Y \arrow[d, "f"] \\ & X \end{tikzcd}$$

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The **Split Short Five Lemma** is precisely the condition that the pullback functor $Pt_X(\mathscr{X}) \to Pt_0(\mathscr{X}) \cong \mathscr{X}$ reflects isomorphisms.

$$\begin{tikzcd}
 & B \arrow[rr, tail] \arrow[dl] \arrow[dd, shift right] & & Z \arrow[dl] \arrow[dd, shift right] \\
A \arrow[rr, tail, crossing over] \arrow[dd, shift right] & & Y & \\
 & 0 \arrow[rr] \arrow[uu, shift right] \arrow[dl, equal] & & X \arrow[uu, shift right] \arrow[dl, equal] \\
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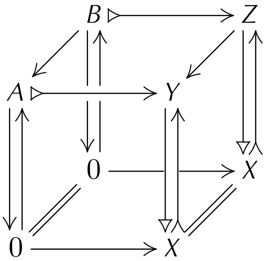

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$$0 \longrightarrow A \rightarrowtail A \rtimes_\xi X \underset{f_\xi}{\overset{s_\xi}{\leftrightarrows}} X \longrightarrow 0.$$

Overview, $n = 1$

	Homology $H_2(X)$	Cohomology $H^2(X,(A,\xi))$	
		trivial action ξ	arbitrary action ξ
Gp	$\dfrac{R \wedge [F,F]}{[R,F]}$	$CentrExt^1(X,A)$	$OpExt^1(X,A,\xi)$
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$$\begin{aligned} f \text{ is central} &\Leftrightarrow \forall_{a\in A}\forall_{e\in E} \quad a = eae^{-1} \\ &\Leftrightarrow \forall_{a\in A}\forall_{e\in E} \quad a = \xi(f(e))(a) \\ &\Leftrightarrow \forall_{x\in X} \quad 1_A = \xi(x) \end{aligned}$$

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How to extend this to semi-abelian categories?

Three commutators

Smith-Pedicchio

For equivalence relations R, S on X

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "\Delta_R" description] \arrow[r, "\Delta_S" description] & S, \arrow[l, shift right=2, "s_2"'] \arrow[l, shift left=2, "s_1"] \end{tikzcd}$$

the **Smith-Pedicchio commutator** $[R,S]^{\mathrm{S}}$ is the kernel pair of t:

$$\begin{tikzcd} & R \arrow[dl, "{\langle 1_R,\Delta_S\circ r_1\rangle}"'] \arrow[d, dotted] \arrow[dr, "r_2"] & \\ R\times_X S \arrow[r, dotted] & T & X \arrow[l, dotted, "t"'] \\ & S \arrow[ul, "{\langle \Delta_R\circ s_1,1_S\rangle}"] \arrow[u, dotted] \arrow[ur, "s_2"'] & \end{tikzcd}$$

Huq & Higgins

For $K, L \lhd X$, the **Huq commutator** $[K,L]^{\mathrm{Q}}$ is the kernel of q:

$$\begin{tikzcd} & K \arrow[dl, "{\langle 1_K,0\rangle}"'] \arrow[d, dotted] \arrow[dr, "k"] & \\ K\times L \arrow[r, dotted] & Q & X \arrow[l, dotted, "q"'] \\ & L \arrow[ul, "{\langle 0,1_L\rangle}"] \arrow[u, dotted] \arrow[ur, "l"'] & \end{tikzcd}$$

The **Higgins commutator** $[K,L] \leqslant X$ is the image of $(k\ l)\circ\iota_{K,L}$:

$$\begin{tikzcd} K\diamond L \arrow[r, "\iota_{K,L}"] \arrow[d, dotted] & K+L \arrow[r] \arrow[d, "(k\ l)"] & K\times L \\ {[K,L]} \arrow[r, dotted, tail] & X & \end{tikzcd}$$

Pregroupoids

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A span $D \xleftarrow{d} X \xrightarrow{c} C$ is a **pregroupoid** iff $[Eq(d), Eq(c)]^{\mathrm{S}} = \Delta_X$. [Kock, 1989]

(β, γ)

$$\begin{tikzcd} & Eq(d) \arrow[dl, "\langle 1_{Eq(d)}, \langle \pi_1, \pi_1 \rangle \rangle"'] \arrow[dr, "\pi_2"] & \\ Eq(d) \times_X Eq(c) \arrow[rr, dotted, "p"] & & X \\ & Eq(c) \arrow[ul, "\langle \langle \pi_1, \pi_1 \rangle, 1_{Eq(c)} \rangle"] \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

(β, α)

$$\begin{tikzcd} & \cdot \arrow[dl, "\beta"'] \arrow[dr, "\gamma"] & \\ \cdot & & \cdot \\ & \cdot \arrow[ul, "\alpha"] \arrow[ur, dotted, "p(\alpha,\beta,\gamma)"'] & \end{tikzcd}$$

$$\begin{cases} p(\alpha, \beta, \beta) = \alpha \\ p(\beta, \beta, \gamma) = \gamma \end{cases}$$

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One (weak and well-studied) such is the ***Smith is Huq* condition (SH)**, which holds when two equivalence relations R and S on an object X commute iff their normalisations $K, L \lhd X$ do.

$$K \rightarrowtail{\!}\xrightarrow{r_2 \circ \ker(r_1)} X \xleftarrow{s_2 \circ \ker(s_1)} L \quad \text{normalisations of} \quad R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X \underset{s_1}{\overset{s_2}{\leftleftarrows}} S$$

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The *Smith is Huq* condition

Several categorical-algebraic conditions have been considered which "make a semi-abelian category behave more like *Gp* does".

One (weak and well-studied) such is the ***Smith is Huq* condition (SH)**, which holds when two equivalence relations R and S on an object X commute iff their normalisations $K, L \lhd X$ do.

$$K \rhd\!\xrightarrow{r_2 \circ \ker(r_1)} X \xleftarrow{s_2 \circ \ker(s_1)}\!\lhd L \quad \text{normalisations of} \quad R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X \overset{s_2}{\underset{s_1}{\leftleftarrows}} S$$

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This is important when defining abelian extensions.

The semi-abelian case: abelian extensions, I

Let $\mathscr{X}$ be a semi-abelian category. An **abelian extension** in $\mathscr{X}$ is a short exact sequence

$$0 \longrightarrow A \rightarrowtail^{a} E \xrightarrow{f} X \longrightarrow 0$$

where f is an **abelian object** in $(\mathscr{X} \downarrow X)$:

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1. the span (f, f) is a pregroupoid;
2. the commutator $[Eq(f), Eq(f)]^{S}$ is trivial;
3. $\langle 1_E, 1_E \rangle \colon E \to Eq(f)$ is a normal monomorphism $f \to f\pi_1$ in $(\mathscr{X} \downarrow X)$;
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Example: a **split extension** (a point (f, s) with $a = \ker(f)$) is abelian iff it is a **Beck module** [Beck, 1967]: an abelian group object in $(\mathscr{X} \downarrow X)$.

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$$\begin{tikzcd}
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0 \arrow[r] & A \arrow[r, "a"'] & E \arrow[r, "f"'] \arrow[u, shift right, "\langle 1_E,1_E \rangle"'] & X \arrow[r] \arrow[u, shift right, "s_\xi"'] & 0
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Given an abelian extension, we may take cokernels as in the diagram on the left to find its **direction**: the X-module (A, ξ).

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The pullback $f^*(\xi)$ of ξ along f is the conjugation action of E on A.

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Theorem (Cohomology with non-trivial coefficients)
$H^2(X, (A, \xi)) \cong OpExt^1(X, A, \xi)$, the group of equivalence classes of extensions from A to X with direction (A, ξ).
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- $H^2(-,(A,\xi))$ is a derived functor of $Hom(-,\ A \rtimes_\xi X \to X) \colon (\mathscr{X} \downarrow X)^{\mathrm{op}} \to Ab$.
  We assume that $\mathscr{X}$ carries a comonad $\mathbb{G}$ whose projectives are the regular projectives.

Overview, $n = 1$

	Homology $H_2(X)$	Cohomology $H^2(X, (A, \xi))$	
		trivial action ξ	arbitrary action ξ
Gp	$\dfrac{R \wedge [F, F]}{[R, F]}$	$CentrExt^1(X, A)$	$OpExt^1(X, A, \xi)$
abelian categories	0	$Ext^1(X, A)$	
Barr-exact categories		$Tors^1[X, (A, \xi)]$	
semi-abelian categories	$\dfrac{R \wedge [F, F]}{[R, F]}$	$CentrExt^1(X, A)$	$OpExt^1(X, A, \xi)$

Overview, arbitrary degrees ($n \geqslant 1$)

	Homology $H_{n+1}(X)$	Cohomology $H^{n+1}(X, (A, \xi))$	
		trivial action ξ	arbitrary action ξ
Gp	$\dfrac{\bigwedge_{i\in n} K_i \wedge [F_n, F_n]}{\bigvee_{I\subseteq n}[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\setminus I} K_i]}$	$CentrExt^n(X, A)$	$OpExt^n(X, A, \xi)$
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Yoneda's extensions

Let X and A be objects in an abelian category $\mathscr{A}$.

A **Yoneda 1-extension** from A to X is a short exact sequence

$$0 \longrightarrow A \rightarrowtail E^1 \xrightarrow{f^1} X \longrightarrow 0.$$

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Theorem [Yoneda, 1960]
If $\mathscr{A}$ has enough projectives, then for $n \geqslant 1$ we have $H^{n+1}(X, A) \cong Ext^n(X, A)$.

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Consider $n \geqslant 2$. A **Yoneda *n*-extension** from A to X is an exact sequence

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How to extend this to semi-abelian categories?

Non-abelian higher extensions: 3^n-diagrams

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0 \wedge K_1 \arrow[r] \arrow[d] & K_0 \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & F_2 \arrow[r, "f_1"] \arrow[d, "f_0"'] & F_{\{1\}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d] & F_{\{0\}} \arrow[r] \arrow[d] & F_{\varnothing} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
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A **double extension** is a 3×3 diagram.

Its rows and columns are short exact sequences.

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0 \arrow[r] & K_0 \wedge K_1 \arrow[r, tail] \arrow[d, tail] & K_0 \arrow[r, two heads] \arrow[d, tail] & \cdot \arrow[r] \arrow[d, tail] & 0 \\
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0 \arrow[r] & K_1 \arrow[r, tail] \arrow[d, two heads] & \color{red}{F_2} \arrow[r, two heads, red, "f_1"] \arrow[d, two heads, red, "f_0"'] & \color{red}{F_{\{1\}}} \arrow[r] \arrow[d, two heads, red] & 0 \\
0 \arrow[r] & \cdot \arrow[r, tail] \arrow[d] & \color{red}{F_{\{0\}}} \arrow[r, two heads, red] \arrow[d] & \color{red}{F_{\varnothing}} \arrow[r] \arrow[d] & 0 \\
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It is determined by an n-fold arrow $F \in Arr^n(\mathscr{X})$,
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0 \arrow[r] & K_0 \wedge K_1 \arrow[r, -{Triangle[open]}] \arrow[d, -{Triangle[open]}] & K_0 \arrow[r, -{Triangle[open]}] \arrow[d, -{Triangle[open]}] & \cdot \arrow[r] \arrow[d, -{Triangle[open]}] & 0 \\
0 \arrow[r] & K_1 \arrow[r, -{Triangle[open]}] \arrow[d, -{Triangle[open]}] & \mathbb{P}_1 \arrow[r, -{Triangle[open]}, "\partial_1"] \arrow[d, -{Triangle[open]}, "\partial_0"'] & \mathbb{P}_0 \arrow[r] \arrow[d, -{Triangle[open]}] & 0 \\
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Example: the n-truncation of any aspherical augmented simplicial object (in particular, any simplicial resolution) determines an $(n+1)$-fold extension (presentation).
In fact, the extension property characterises being aspherical [Everaert, Goedecke & VdL, 2012].

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A **double extension** is a 3×3 diagram.
Its rows and columns are short exact sequences.

The red square $F \in Arr^2(\mathscr{X})$ which determines it is a **regular pushout**: its arrows and the comparison $F_2 \to F_{\{0\}} \times_{F_\varnothing} F_{\{1\}}$ are regular epimorphisms.

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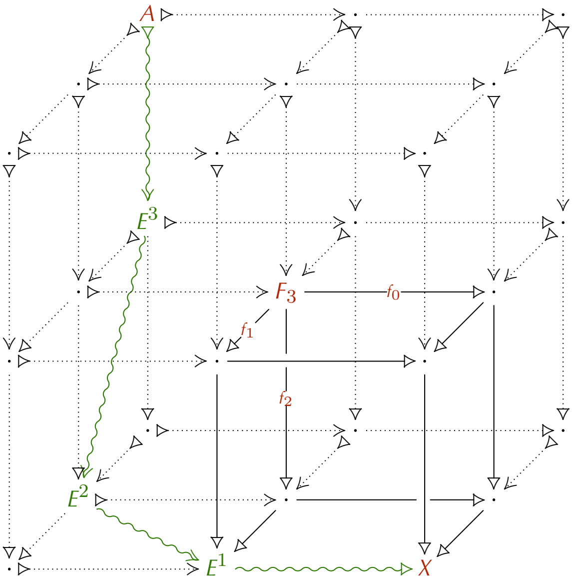
In the abelian case, Yoneda *n*-extensions are equivalent to *n*-fold extensions (by Dold-Kan).

Abelian case: 3-fold extension vs. Yoneda 3-extension



Overview, arbitrary degrees ($n \geqslant 1$)

	Homology $H_{n+1}(X)$	Cohomology $H^{n+1}(X,(A,\xi))$	
		trivial action ξ	arbitrary action ξ
Gp	$\dfrac{\bigwedge_{i\in n} K_i \wedge [F_n, F_n]}{\bigvee_{I\subseteq n}[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\setminus I} K_i]}$	$CentrExt^n(X,A)$	$OpExt^n(X,A,\xi)$
abelian categories	0	$Ext^n(X,A)$	
Barr-exact categories		$Tors^n[X,(A,\xi)]$	
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Repeating this construction gives a definition of *n-fold central extensions* for all *n*.

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- F is an *n-fold projective presentation*; its "initial maps" $f_i\colon F_n \to F_{n\setminus\{i\}}$ have kernel K_i.
- The object $L_n[F]$ is what must be divided out of F_n to make F central.
- By [Rodelo & VdL, 2012], under (SH), the object $L_n[F]$ is a join $\bigvee_{I\subseteq n}\Big[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\setminus I} K_i\Big]$ as in [Brown & Ellis, 1988] [Donadze, Inassaridze & Porter, 2005].
- In fact, the Hopf formula is valid for any Birkhoff reflector $I\colon \mathscr{X} \to \mathscr{Y}$.
- Alternatively, $H_{n+1}(X, I) \cong \lim(\mathit{CExt}^n_{I,X}(\mathscr{X}) \to \mathscr{Y} : F \mapsto \bigwedge_{i\in n} K_i)$. [Goedecke & VdL, 2009]

Overview, arbitrary degrees ($n \geqslant 1$)

	Homology $H_{n+1}(X)$	Cohomology $H^{n+1}(X,(A,\xi))$	
		trivial action ξ	arbitrary action ξ
Gp	$\dfrac{\bigwedge_{i\in n} K_i \wedge [F_n,F_n]}{\bigvee_{I\subseteq n}[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\setminus I} K_i]}$	$CentrExt^n(X,A)$	$OpExt^n(X,A,\xi)$
abelian categories	0	$Ext^n(X,A)$	
Barr-exact categories		$Tors^n[X,(A,\xi)]$	
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Cohomology classifies higher central extensions

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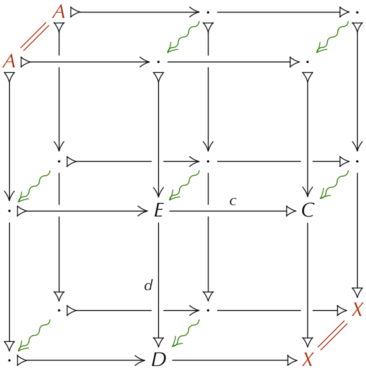

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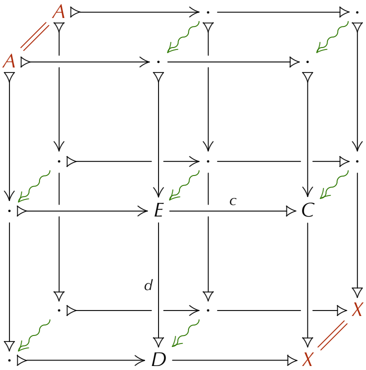

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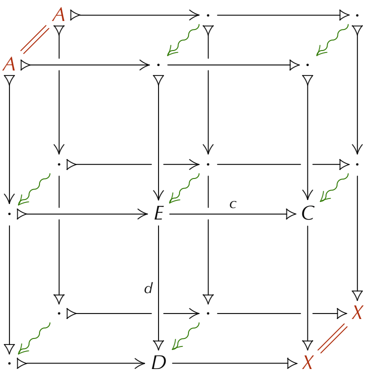

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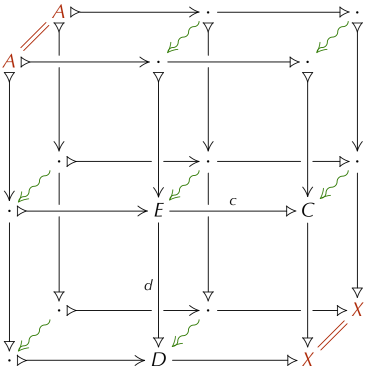

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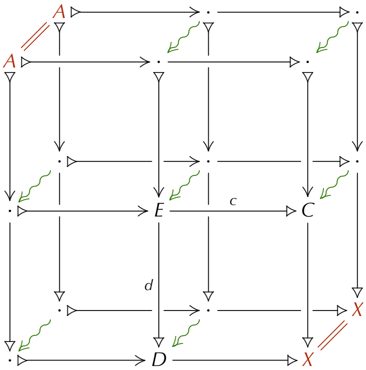

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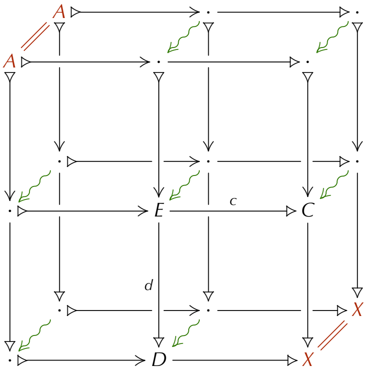

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  (T1) $\mathbb{t}$ is a fibration which is exact from degree n on;
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If (A, ξ) is a trivial X-module in a semi-abelian category with (SH), then (1) any torsor, viewed as an n-extension, is central; and (2) every class in $CentrExt^n(X, A)$ contains a torsor.

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[Peschke, Simeu & VdL, work-in-progress]

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n-pregroupoid condition

An n-fold analogue $[Eq(f_0), \ldots, Eq(f_{n-1})]^{\mathrm{S}}$ of the Smith commutator of the $Eq(f_i)$ is trivial $\rightsquigarrow$ *higher-order Mal'tsev operation*

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K_1 \arrow[r, tail] \arrow[d, two heads] & F_2 \arrow[r, two heads, "f_1"] \arrow[d, two heads, "f_0"'] & \cdot \arrow[d, two heads] \\
\cdot \arrow[r, tail] & \cdot \arrow[r, two heads] & X
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If (A, ξ) is a trivial X-module, then an n-extension from A to X is connected to a torsor over $\mathbb{K}((A, \xi), n)$ iff it is central.

- When $n = 2$ this means that $[K_0, K_1] = 0 = [A, F_2]$.

The case of non-trivial coefficients is much harder, because here the proof techniques by induction of categorical Galois theory are no longer available.

Question: When is an n-extension connected to an (A, ξ)-torsor?
Answer: When it is an n-pregroupoid with direction (A, ξ).

An n-extension is in a class in $OpExt^n(X, A, \xi)$ iff it satisfies the following two conditions:

n-pregroupoid condition

An n-fold analogue $[Eq(f_0), \ldots, Eq(f_{n-1})]^{S}$ of the Smith commutator of the $Eq(f_i)$ is trivial $\rightsquigarrow$ *higher-order Mal'tsev operation*

direction is (A, ξ)

The pullback $(F_n \to X)^*(\xi)$ of ξ is the conjugation action of F_n on A.

Non-trivial coefficients

[Peschke, Simeu & VdL, work-in-progress]

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Is it $\bigvee_{\varnothing \neq I \subsetneq n} [\bigwedge_{i \in I} K_i, \bigwedge_{i \in n \setminus I} K_i] = 0$?

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Under (SH), any n-extension from A to X has a direction which is an X-module (A, ξ).

Some final remarks

- For a complete picture of cohomology with non-trivial coefficients, mainly certain aspects of commutator theory need to be further developed: in particular, higher Smith commutators, and their decomposition into (potentially non-binary) Higgins commutators.

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  We made heavy use of the condition (SH), but a whole hierarchy of categorical-algebraic conditions has been introduced and studied over the last few years: some examples are (local) algebraic cartesian closedness, action representability, action accessibility, algebraic coherence, strong protomodularity, normality of Higgins commutators.

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- These categorical conditions may help us understand algebra from a new perspective. For instance, they might lead to a categorical characterisation of Gp, $\mathit{Lie}_{\mathbb{K}}$, etc.

Coda

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, tail] \arrow[d, tail] & \cdot \arrow[r, two heads] \arrow[d, tail] & \cdot \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & \cdot \arrow[r, tail] \arrow[d, two heads] & \cdot \arrow[r, two heads] \arrow[d, two heads] & \cdot \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \cdot \arrow[r, tail] \arrow[d] & \cdot \arrow[r, two heads] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Higher central extensions play “dual” roles
in the interpretation of homology and cohomology
(with trivial coefficients):

Coda

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 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & {\color{red}A} \arrow[r, tail] \arrow[d, tail] & \cdot \arrow[r, two heads] \arrow[d, tail] & \cdot \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & \cdot \arrow[r, tail] \arrow[d, two heads] & \cdot \arrow[r, two heads] \arrow[d, two heads] & \cdot \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \cdot \arrow[r, tail] \arrow[d] & \cdot \arrow[r, two heads] \arrow[d] & {\color{red}X} \arrow[r] \arrow[d] & 0 \\
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Theorem [Peschke & VdL, 2016]
If X is a group and $n \geqslant 1$, then $H_{n+1}(X) \cong Hom(H^{n+1}(X, -), 1_{Ab})$.

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- This may also be shown via a *non-additive derived Yoneda lemma*.

Thank you!