Fibred Representation of Linear Structure

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Joint work with Robin Cockett and Jonathan Gallagher

Overview

- Categorical quantum mechanics has shown that compact closed dagger categories provide an abstract framework to develop many concepts in quantum physics.
- Using a minimal axiomatic scheme can clarify structure.

Overview

- Categorical quantum mechanics has shown that compact closed dagger categories provide an abstract framework to develop many concepts in quantum physics.
- Using a minimal axiomatic scheme can clarify structure.
- I've been studying classical mechanics - Hamiltonian and Lagrangian mechanics - in order to formalize those structures in a tangent category.
- In this talk, we're going to explore the properties of vector bundles in the category of smooth manifolds in order to capture them in an abstract fibration.

Vector Bundles

A *smooth* $\mathbb{R}$*-vector bundle* is epimorphism $E \xrightarrow{q} M$ and real vector space V in the category of smooth manifolds such that:

$$
\begin{array}{ccc}
E \times_M E & \xrightarrow{\;+\;} & E \\
& \searrow \quad \swarrow & \\
& M &
\end{array}
\qquad
\begin{array}{ccc}
E \times \mathbb{R} & \xrightarrow{\;\cdot\;} & E \\
& \searrow \quad \swarrow & \\
& M &
\end{array}
$$

Such that for every point $m \in M$ there exists $U \subseteq M, m \in U$ such that

$$q^{-1}(U) \cong U \times V$$

Remark: The pullback of a vector bundle is a vector bundle!

The tangent bundle

The canonical example of a vector bundle is the *tangent bundle* of a smooth manifold M, $T(M)$.

$T(M)$: equivalence classes of curves $\mathbb{R} \longrightarrow M$

$p : T(M) \longrightarrow M$ is evaluation at 0.

Phase Space and the Cotangent Bundle

Configuration space: The possible states of a physical system. Each *configuration* - a valid set of parameters - is a point on a manifold M.

Phase space: All possible *configuration* and *momentum* values for a physical system.
A momentum value is a map $T(M) \longrightarrow \mathbb{R}$, otherwise known as a *cotangent vector*.

The phase space is the *cotangent bundle* of M, $p_M^* : T^*(M) \longrightarrow M$.

The Vector Bundle Fibration

Consider two fibrations on the category of smooth manifolds:

$$\begin{array}{ccc} VLin & \hookrightarrow & VBun \\ & \searrow & \downarrow \\ & & SMan \end{array}$$

VBun: Full subcategory of $SMan^{\rightarrow}$ whose objects are vector bundles.

VLin: The subfibration of **VBun** restricted to linear bundle morphisms.

Some Issues

Cockett and Cruttwell showed that the fibres of "**VBun**" in a "nice" tangent category admit the logic of calculus. However, it's missing many of the structures used in mechanics!

- Tensor product of bundles and linear maps.
- Dual bundles
- R-module structure

In order to characterize these structures abstractly, we use the machinery in:

- *Cartesian Differential Storage Categories*, Blute, Cockett and Seely.
- *Duality and Traces for Indexed Monoidal Categories*, Ponto and Shulman.
- *Categorical Models of PiLL*, Birkedal, Møgelberg, and Peterson.

Simple Fibration

Suppose $\partial : \mathbb{E} \longrightarrow \mathbb{B}$ is a fibration with finite fibred products. Define the **simple fibration above** ∂ (Jacobs) $\pi : \mathbb{E}[\partial] \longrightarrow \mathbb{E}$ as follows

- Objects: (I, X) in $\mathbb{E} \underset{\mathbb{B}}{\times} \mathbb{E}$
- Maps: $$\frac{(u, f) : (I, X) \longrightarrow (J, Y) \quad \mathbb{E}[\partial]}{(u, f) : (I, I \underset{A}{\times} X) \longrightarrow (J, Y) \quad \mathbb{E} \underset{\mathbb{B}}{\times} \mathbb{E}}$$
- Cartesian maps:

$$\begin{array}{ccc} \mathbb{E}[\partial] & (I, \partial(u)^*(Y)) & \xrightarrow{(u, \pi_1^A \partial(u)^*_Y)} & (J, Y) \\ \downarrow\pi & & & \\ \mathbb{E} & I & \xrightarrow{u} & J \end{array}$$

Fibred System of Linear Maps

A system of linear maps $\pi_L : \mathbb{L} \longrightarrow \mathbb{E}$ above a fibration ∂ is a fibration

$$\begin{array}{ccc} \mathbb{L} & \stackrel{L}{\hookrightarrow} & \mathbb{E}[\partial] \\ & {\scriptstyle \pi_L} \searrow & \downarrow \pi \\ & & \mathbb{E} \end{array}$$

Such that

- L is a bijection on objects
- L is a fibred product preserving subfibration

Linear maps

Linear in an argument:

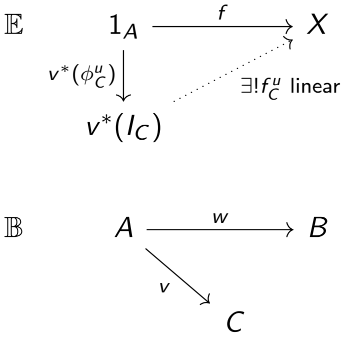
$$\frac{(j, f) : (I, X) \longrightarrow (J, Y) \quad \text{in } \mathbb{L}}{f : I \underset{A}{\times} X \longrightarrow Y \in \mathbb{E} \text{ is } \textit{linear in } X}$$

There is a fibration ∂_L of linear maps above $\mathbb{B}$ which is induced by pullback of fibrations:

$$\begin{array}{ccc} Lin & \longrightarrow & \mathbb{L} \\ {\scriptstyle \partial_L}\downarrow & & \downarrow{\scriptstyle \pi_L} \\ \mathbb{B} & \underset{!}{\longrightarrow} & \mathbb{E} \end{array}$$

Unit Representation

A system of linear maps $\mathbb{L}$ over $\partial : \mathbb{E} \longrightarrow \mathbb{B}$ has *representable unit* when:



Strong Unit Representation

A system of linear maps $\mathbb{L}$ over $\partial : \mathbb{E} \longrightarrow \mathbb{B}$ has **strong** unit representation when for every

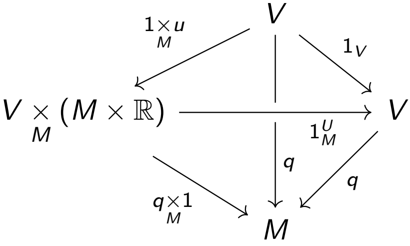
$$\begin{array}{ccc} Z \underset{A}{\times} v^*(Y) \underset{A}{\times} 1_A & \xrightarrow{\ f\ } & X \\ {\scriptstyle 1 \underset{A}{\times} 1 \underset{A}{\times} v^*(\phi^u_C)} \Big\downarrow & \nearrow {\scriptstyle \exists! f^u_C \text{ linear in } I_C} & \\ Z \underset{A}{\times} v^*(Y \underset{C}{\times} I_C) & & \end{array}$$

Persistent unit representation:

$$\frac{f : Z \underset{A}{\times} 1_A \longrightarrow X \quad \text{linear in Z}}{f^U_C : Z \underset{A}{\times} v^*(I_C) \longrightarrow X \text{ linear in Z}}$$

Example: Smooth Manifolds

Scalar multiplication arises from unit representation.



- $u(m) = (m, 1) \in M \times \mathbb{R}$
- $1^U_M(v, (m, r)) = (m, r) \cdot v$

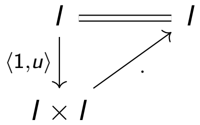
Theorem

Given a system of linear maps $\pi_L : \mathbb{L} \longrightarrow \mathbb{E}$ *over* $\partial : \mathbb{E} \longrightarrow \mathbb{B}$ *with strong and persistent unit representation*

1. *There is a morphism of fibrations* $I : 1_{\mathbb{B}} \longrightarrow \partial$
2. *I sends each object of A to a commutative monoid object in the fiber category above A whose multiplication is bilinear.*

Proof of 2

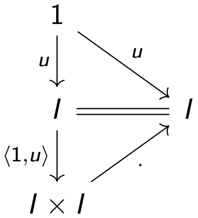
Define multiplication to be the unique map:



By persistence, $\cdot$ is bilinear.

Proof of 2

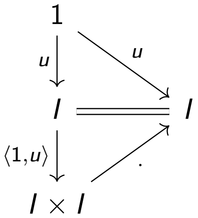
Define multiplication to be the unique map:



Note that 1_I also has a universal property

Proof of 2

Define multiplication to be the unique map:



Note that 1_I also has a universal property

Thus, $\cdot$ is the unique map such that $u\langle 1, !u\rangle \cdot = u$.

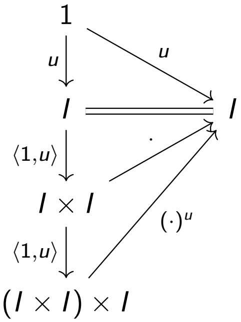
Multiplication is symmetric

It follows that $\cdot$ is symmetric:

$$
\begin{aligned}
u\langle 1, !u\rangle\tau\cdot &= \langle u, u\rangle\tau\cdot \\
&= \langle u, u\rangle\cdot \\
&= u\langle 1, !u\rangle\cdot \\
&= u
\end{aligned}
$$

Multiplication is associative

Induce another map via universal property:



Then observe that $(\cdot)^u = (1 \times \cdot)\cdot = (\cdot \times 1)\cdot$

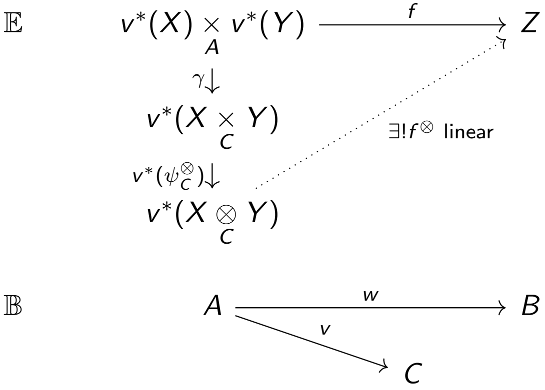
$$\begin{aligned} u\langle 1, !u\rangle\langle 1, !u\rangle(1 \times \cdot)\cdot =& \langle u, u, u\rangle(1 \times \cdot)\cdot \\ =& \langle u, u\rangle\cdot \\ =& u \end{aligned}$$

And:

$$\begin{aligned} u\langle 1, !u\rangle\langle 1, !u\rangle(\cdot \times 1)\cdot =& \langle u, u, u\rangle(\cdot \times 1)\cdot \\ =& \langle u, u\rangle\cdot \\ =& u \end{aligned}$$

Tensor Representation

A system of linear maps on a fibration $\partial : \mathbb{E} \longrightarrow \mathbb{B}$ has a *representable tensor* whenever for any bilinear map f:



Strong Tensor Representation

$$\begin{array}{ccc}
W \underset{A}{\times} v^*(X) \underset{A}{\times} v^*(Y) & \xrightarrow{\quad f \quad} & Z \\
\gamma\downarrow & & \\
W \underset{A}{\times} v^*(X \underset{C}{\times} Y) & & \exists! f^{\otimes} \text{ linear in } v^*(X \underset{C}{\otimes} Y) \\
v^*(\psi_C^{\otimes})\downarrow & & \\
W \underset{A}{\times} v^*(X \underset{C}{\otimes} Y) & \cdots\cdots\cdots\cdots\nearrow &
\end{array}$$

Persistence:

$$\frac{f : W \underset{A}{\times} v^*(X) \underset{A}{\times} v^*(Y) \longrightarrow Z \quad \text{linear in W}}{f^{\otimes} : W \underset{A}{\times} v^*(X \underset{C}{\otimes} Y) \longrightarrow Z \text{ linear in W}}$$

Tensor representation of vector bundles:

Over each fibre of a bilinear map $(f, u) : q_1 \underset{M}{\times} q_2 \longrightarrow q_3$ restricts to a bilinear morphism of vector spaces:

$$\begin{array}{ccc} q_1^{-1}(m) \times q_2^{-1}(m) & \xrightarrow{f} & q_3^{-1}(u(m)) \\ \downarrow \psi^{\otimes} & \nearrow \exists! f^{\otimes} & \\ q_1^{-1}(m) \otimes q_2^{-1}(m) & & \end{array}$$

Theorem

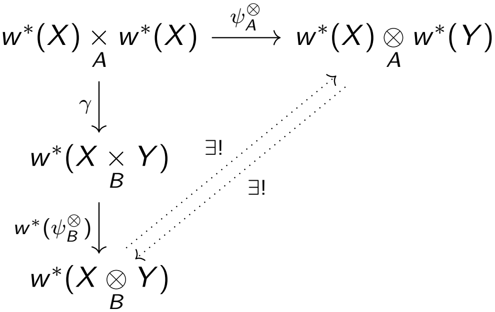
If system of linear maps π_L has strong and persistant unit and tensor representation then ∂_L is a fibred symmetric monoidal category.

Need only show that $\otimes$ is a morphism of fibrations, the rest of the proof can be lifted from Blute-Cockett-Seely.
First, define $f \underset{w}{\otimes} g$:

$$
\begin{array}{ccc}
W \underset{A}{\times} X & \xrightarrow{f \underset{w}{\times} g} & Y \underset{B}{\times} Z \\
\downarrow{\scriptstyle \psi^{\otimes}} & & \downarrow{\scriptstyle \psi^{\otimes}} \\
W \underset{A}{\otimes} X & \underset{\exists ! f \underset{w}{\otimes} g}{\dashrightarrow} & Y \underset{B}{\otimes} Z
\end{array}
$$

Proof of 1

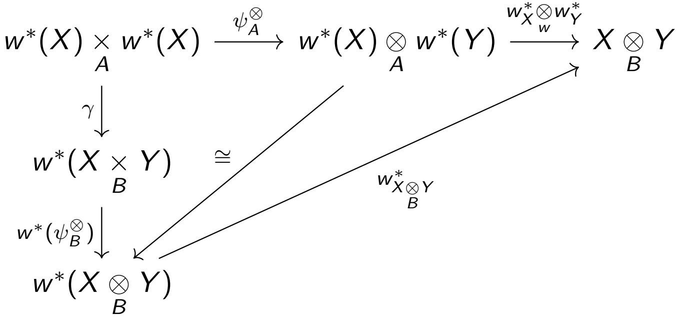
First, note:



Thus we have an isomorphism $w^*(X \underset{B}{\otimes} Y) \cong w^*(X) \underset{A}{\otimes} w^*(Y)$.

Proof of 1

$$w^*(X) \xrightarrow{w_X^*} X, w^*(Y) \xrightarrow{w_Y^*} Y \text{ cartesian above } A \xrightarrow{w} B:$$



Representable Hom

A linear system of maps has a representable hom $\_ \multimap \_$ if for every map f linear in $v^*(X)$

$$
\begin{array}{ccc}
 & & v^*(X \multimap Y) \underset{B}{\times} v^*(X) \\
 & \exists! \lambda(f) \underset{w}{\times} w^*_X \nearrow & \downarrow \gamma \\
 & & v^*(X \multimap Y \underset{B}{\times} X) \\
 & & \downarrow v^*(ev) \text{ bilinear} \\
W \underset{A}{\times} w^*(v^*(X)) & \xrightarrow{\quad f \quad} & v^*(Y)
\end{array}
$$

$$A \xrightarrow{\quad w \quad} B \xrightarrow{\quad v \quad} C$$

Persistent:

$$\frac{f : X \times Y \to Z \quad \text{bilinear}}{\lambda f : X \to Y \multimap Z \ \text{linear}}$$

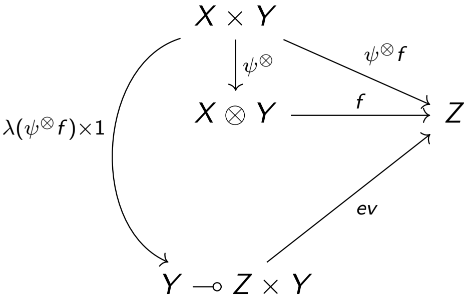
Theorem

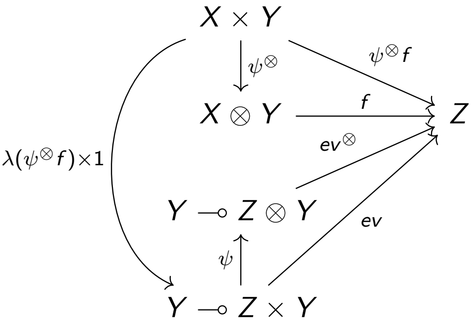
Given a linear system of maps π_L *of* $\mathbb{E}[\partial] \xrightarrow{\pi} \mathbb{E}$

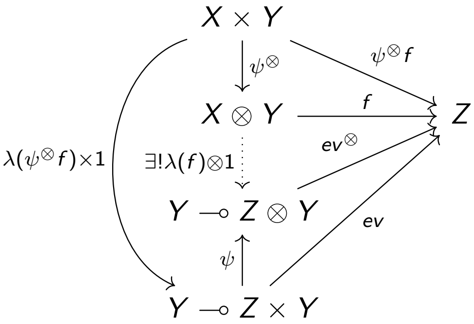
1. *If* π_L *has a strong persistent representable unit and persistent representable hom then* ∂_L *is a fibred closed category.*
2. *If* π_L *has a strong persistent representable unit and tensor, and a persistent representable hom then* ∂_L *is a fibred symmetric monoidal closed category .*

$$X \otimes Y \xrightarrow{f} Z$$

$$
\begin{array}{ccc}
X \times Y & & \\
\downarrow{\scriptstyle \psi^{\otimes}} & \searrow^{\psi^{\otimes} f} & \\
X \otimes Y & \xrightarrow{f} & Z
\end{array}
$$


$X \times Y$
$\psi^{\otimes}$
$\psi^{\otimes} f$
$X \otimes Y$
f
Z
$\lambda(\psi^{\otimes} f) \times 1$
ev
$Y \multimap Z \times Y$


$X \times Y$
$\psi^{\otimes}$
$\psi^{\otimes} f$
$X \otimes Y$
f
Z
$ev^{\otimes}$
$\lambda(\psi^{\otimes} f) \times 1$
$Y \multimap Z \otimes Y$
ev
ψ
$Y \multimap Z \times Y$


$X \times Y$
$\psi^{\otimes}$
$\psi^{\otimes} f$
$X \otimes Y$
f
Z
$ev^{\otimes}$
$\lambda(\psi^{\otimes} f) \times 1$
$\exists! \lambda(f) \otimes 1$
$Y \multimap Z \otimes Y$
ev
ψ
$Y \multimap Z \times Y$

Future Work

- Develop symplectic geometry in this setting
  - Momentum maps and Noether's theorem
- The linear hom in a type system
- Expand this to include *storage*
- Develop a graphical calculus

Thank You.