DEPARTAMENTO DE MATEMÁTICAS

A characterisation of Lie algebras amongst alternating algebras

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Joint work with Tim Van der Linden

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Definition

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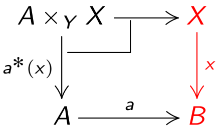

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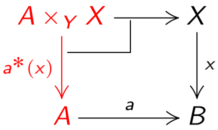

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A category $\mathcal{C}$ is locally cartesian closed if all functors a^* have a right adjoint.

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Let $\mathcal{C}$ be a category. For a fixed object $B \in \mathcal{C}$, we form the category of points over B, denoted by $\mathtt{Pt}_B(\mathcal{C})$.

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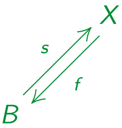

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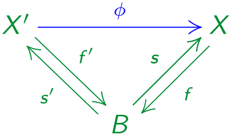

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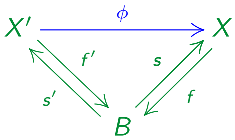

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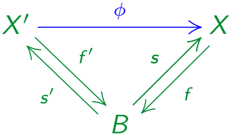

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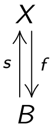

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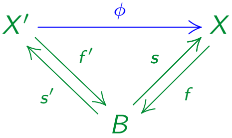

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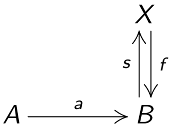

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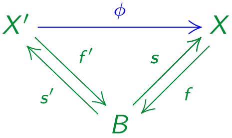

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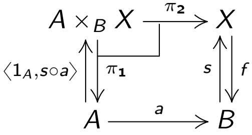

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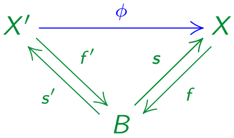

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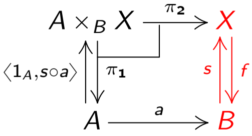

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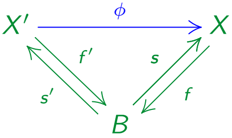

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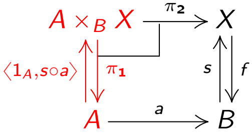

Protomodularity

Definition (Bourn, 1991)

A category $\mathcal{C}$ is protomodular if all $a^* \colon \mathtt{Pt}_B(\mathcal{C}) \to \mathtt{Pt}_A(\mathcal{C})$ reflect isomorphisms.

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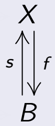

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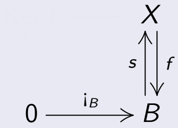

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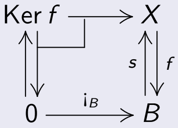

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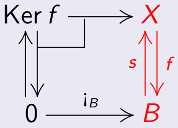

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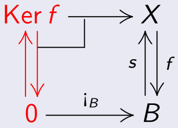

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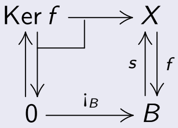

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A category $\mathcal{C}$ is locally algebraically cartesian closed (or (LACC) for short) if all $a^* \colon \mathrm{Pt}_B(\mathcal{C}) \to \mathrm{Pt}_A(\mathcal{C})$ have a right adjoint.

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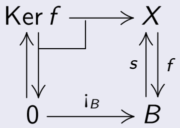

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Non-associative algebras

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Examples

Lie algebras, Associative algebras, Jordan algebras, Alternating algebras, the abelian case...

Non-associative algebras

Theorem (Zhevlakov, Slin'ko, Shestakov, Shirshov)

If $\mathcal{V}$ is a variety of algebras over an infinte field $\mathbb{K}$, all of its laws are of the form $\phi(x_1, \ldots, x_n)$, where ϕ is a non-associative polynomial.
Moreover, each of its homogeneous components $\psi(x_{i_1}, \ldots, x_{i_n})$ is also a law.

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Constructions in varieties algebras (informally)

Let $\mathcal{V}$ be a variety of algebras, the free $\mathcal{V}$-algebra generated by a set $X = \{x, y, z, \dots\}$ is the $\mathbb{K}$-vector space with basis the free magma generated by X

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The coproduct in $\mathcal{V}$ of two algebras A and B, is the free algebra generated by A and B where if we find a word formed by just elements from one of them, we substitute by its multiplication.

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Theorem (Bourn-Janelidze, 1998)

There is an equivalence of categories

$$\mathtt{Pt}_B(\mathcal{V}) \simeq B\text{-}\mathtt{Act}(\mathcal{V})$$

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$$\begin{array}{c} B\flat X \\ \big\downarrow \\ X \end{array} \longmapsto X$$

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(LACC)

Proposition (Gray, 2012)

The following are equivalent:

- $\mathcal{V}$ *is (LACC).*
- $i_B^* \colon B\text{-}\mathtt{Act}(\mathcal{V}) \to \mathcal{V}$ *preserves binary sums.*
- *The cannonical comparison* $(B\flat X + B\flat Y) \to B\flat(X + Y)$ *is an isomorphism.*

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But it also comes from $x(yb_2) \in B\flat(X + Y)$.

Since yb_2 plays the role of "one element" in $B\flat Y$, either yb_2 is zero, or x multiplied by something from $B\flat Y$ is zero. In both cases it implies that the algebra is abelian.

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$$(B\flat X + B\flat Y) \to B\flat(X + Y)$$

Then $(xb)y$ and $x(by)$ go to the same element in $B\flat(X + Y)$ but they are different in $(B\flat X + B\flat Y)$.

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Then, we have that $(bx)y + (xb)y = 0$.
Again, $(bx)y + (xb)y$ is zero in $B\flat(X + Y)$ but it does not need to be in $(B\flat X + B\flat Y)$.

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Theorem (G.M.-Van der Linden, 2017)

Let $\mathcal{V}$ a variety of non-associative algebras. Then, the following are equivalent:

- *$\mathcal{V}$ is algebraic coherent, i.e. the map $(B\flat X + B\flat Y) \to B\flat(X+Y)$ is a regular epimorphism (Cigoli-Gray-Van der Linden, 2015).*

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