On a problem in Objective Number Theory

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“Diophantus probably considered natural numbers not in the abstract way which we habitually now do, but as born from actual objects. While the method of formally adjoining negatives and ascending to powerful cohomological calculations etc. leads to many results, we should not forget the objects themselves. Just as realizing cohomology classes by vector bundles via K-theory permitted powerful interplay between those calculations and directed manipulations of the objects by actual maps, so a similar possibility is opened by the Burnside rig of a distributive category, wherein polynomial equations satisfied by objects are revealed as specific structure on the objects themselves. […]”

A category with finite products and coproducts is said to be **distributive** if the canonical maps

\[ \sum_{i=1}^{n} (A_i \times B) \rightarrow \left( \sum_{i=1}^{n} A_i \right) \times B \]

are isomorphisms; here \( n \geq 0 \), but the case \( n = 2 \) suffices [...]

A category is **preextensive** if it is extensive and has finite products. 

**Proposition** Prextensive categories are distributive.
Distributive and extensive categories

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are isomorphisms; here \( n \geq 0 \), but the case \( n = 2 \) suffices [...] A category \( \mathcal{E} \) **extensive** if it has finite coproducts and for each pair \( A, B \) of objects the obvious functor

\[
\mathcal{E}/A \times \mathcal{E}/B \rightarrow \mathcal{E}/A + B
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is an equivalence. (It follows that \( \mathcal{E}/0 \rightarrow \mathcal{E} \) is an equivalence.)
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**Proposition**
Distributive and extensive categories

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A category \( E \) **extensive** if it has finite coproducts and for each pair \( A, B \) of objects the obvious functor

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A category is **preextensive** if it is extensive and has finite products.

**Proposition**

Preextensive categories are distributive.
Definition ("ring without negatives" (Schanuel’91))

A rig is a pair commutative monoids \((A, \cdot, 1)\) and \((A, +, 0)\) with the same underlying object and such that:

\[
0 = a \cdot 0 \\
(a \cdot b) + (a \cdot c) = a \cdot (b + c)
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Example (Rigs)

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Rigs

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Example (Rigs)

1. Rings and Distributive lattices.
2. \(\mathbb{N}, \mathbb{Q}_+, \mathbb{R}_+\).
3. The Burnside rig of isomorphism classes of objects in a preextensive category. (The additive structure never is a non-trivial group under.)

The Burnside rig of a distributive \(\mathbf{E}\) will be denoted by \(\mathcal{B}\mathbf{E}\).
Negative sets have Euler characteristic and dimension

“1. Where are the negative sets?
Though ill-posed, the question is suggestive; [...] we seek an enlargement $\mathbb{P}$ [of the category of finite sets], the isomorphism classes of which should give rise to the integers, rather than just natural numbers.”
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Why ill posed?
Let $\mathbb{P}_0$ be the (prextensive) category of bounded polyhedra.
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Why ill posed?

Let $\mathbb{P}_0$ be the (preextensive) category of bounded polyhedra.

The obvious

$$ (0, 1) + 1 + (0, 1) \xrightarrow{\cong} (0, 1/2) + \{1/2\} + (1/2, 1) \xrightarrow{} (0, 1) $$

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Theorem (Schanuel’91)

The induced $\mathbb{N}[X]/(X = 2X + 1) \rightarrow \mathcal{B}(\mathbb{P}_0)$ is an isomorphism.

Proof)
Negative sets have Euler characteristic and dimension

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Though ill-posed, the question is suggestive; [...] we seek an enlargement \( \mathbb{E} \) [of the category of finite sets], the isomorphism classes of which should give rise to the integers, rather than just natural numbers.”

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\]
is an isomorphism.

Theorem (Schanuel’91)

*The induced* \( \mathbb{N}[X]/(X = 2X + 1) \rightarrow \mathcal{B}(\mathbb{P}_0) \) *is an isomorphism.*

Proof) “Surjectivity is easy, because every bounded polyhedron is a disjoint union of open simplexes [...]. The heart of the matter is the injectivity of our map”
“While the study of linear equations on distributive categories is packed with surprising subtleties, higher-degree equations are also approachable […] 

I was surprised to note that an isomorphism \( x = 1 + x^2 \) (leading to complex numbers as Euler characteristics if they don't collapse) always induces an isomorphism \( x^7 = x \).”
Blass and Gates

Theorem (Blass’95)

There exists a prextensive $E$ such that $\mathcal{B}E \cong \mathbb{N}[X]/(X = 1 + X^2)$.
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Theorem (Gates’98)

For \( L(X) \in \mathbb{N}[X] \) having at least one constant term and at least one nonconstant term, there exists a prextensive category \( E \) such that \( \mathcal{B}E \cong \mathbb{N}[X]/(X = L(X)) \).

Proof.
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Theorem (Gates’98)

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Proof.

The hypotheses imply the existence of a faithful $\mathcal{E} \rightarrow \text{Set}$.
Theorem

For any $L(X) \in \mathbb{N}[X]$ there exists a prextensive category $E$ such that $\mathcal{B}E \cong \mathbb{N}[X]/(X = L(X))$.

The case of constant polynomials is trivial.
A preextensive $E$ and a surjective $\mathbb{N}[X]/(X = L(X)) \to \mathfrak{B}E$.
(Using the strategy in Blass’ paper.)
Finite products and families

Let $\mathcal{T}$ be a small category with finite products equipped with a Grothendieck topology $J$. 

Let $\text{Fam}_\mathcal{T}$ be the finite-coproduct completion of $\mathcal{T}$. (It is prextensive. The induced $\text{Fam}_\mathcal{T} \to \hat{\mathcal{T}}$ preserves finite products and finite coproducts.)

Define $\text{Fam}(\mathcal{T}, J) \to \text{Sh}(\mathcal{T}, J)$ as the image of the top-right composite below $\text{Fam}_\mathcal{T} \downarrow \downarrow \to \hat{\mathcal{T}} \downarrow \downarrow \text{Fam}(\mathcal{T}, J) \to \text{Sh}(\mathcal{T}, J)$ so that $\text{Fam}(\mathcal{T}, J)$ is prextensive and $\text{Fam}(\mathcal{T}, J) \to \text{Sh}(\mathcal{T}, J)$ is a full inclusion preserving finite products and finite coproducts.
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\[
\begin{array}{ccc}
\text{Fam}\mathcal{T} & \xrightarrow{a} & \hat{\mathcal{T}} \\
\downarrow & & \downarrow \\
\text{Fam}(\mathcal{T}, J) & \xrightarrow{a} & \text{Sh}(\mathcal{T}, J)
\end{array}
\]

so that
Let $\mathcal{T}$ be a small category with finite products equipped with a Grothendieck topology $J$. Let $\text{Fam}\mathcal{T}$ be the finite-coproduct completion of $\mathcal{T}$. (It is prextensive. The induced $\text{Fam}\mathcal{T} \to \mathcal{T}$ preserves finite products and finite coproducts.)

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\[
\begin{array}{ccc}
\text{Fam}\mathcal{T} & \longrightarrow & \mathcal{T} \\
\downarrow & & \downarrow^a \\
\text{Fam}(\mathcal{T}, J) & \longrightarrow & \text{Sh}(\mathcal{T}, J)
\end{array}
\]

so that $\text{Fam}(\mathcal{T}, J)$ is prextensive and $\text{Fam}(\mathcal{T}, J) \to \text{Sh}(\mathcal{T}, J)$ is a full inclusion preserving finite products and finite coproducts.
(Free) Algebraic theories

\[
\begin{align*}
\text{Theories} & \xrightarrow{\text{free}} \text{Set}^\mathbb{N} \\
& \xrightarrow{\text{forget}} \text{‘signatures’}
\end{align*}
\]

Example

The polynomial \(3X^2\) ∈ \(\mathbb{N}[X]\) induces a ‘signature’ presenting a theory with 3 operations of arity 2, and no equations.

In general: \(L(X) \in \mathbb{N}[X] \rightarrow \ell \in \text{Set}^{\mathbb{N}\text{free}} \rightarrow \ell \in \text{Theories}\)
(Free) Algebraic theories

\[ \text{Theories} \xrightarrow{\text{free}} \xrightarrow{\text{forget}} \text{Set}^\mathbb{N} \]
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The polynomial \(3X^2 \in \mathbb{N}[X]\) induces a ‘signature’ presenting a theory with 3 operations of arity 2, and no equations.

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L(X) \in \mathbb{N}[X] \rightarrow \ell \in \text{Set}^\mathbb{N} \xrightarrow{\text{free}} \mathcal{L} \in \text{Theories}
\]
A coverage on a free algebraic theory

\[
\begin{array}{ccc}
\text{Set}^\mathbb{N} & \xrightarrow{\text{free}} & \text{Theories} \\
\downarrow & \rotatebox{90}{\shortrightarrow} & \downarrow \\
& \text{forget} & \\
L(X) \in \mathbb{N}[X] & \longrightarrow & \ell \in \text{Set}^\mathbb{N} & \xrightarrow{\text{free}} & \mathcal{L} \in \text{Theories}
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Denote the objects of \( \mathcal{L} \) by \( T^A \) with \( A \) a finite set.
A coverage on a free algebraic theory

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& \xleftarrow{\text{forget}} \text{Set}^\mathbb{N} \\
L(X) \in \mathbb{N}[X] & \xrightarrow{\ell} \ell \in \text{Set}^\mathbb{N} & \text{free} & \xrightarrow{\ell} \mathcal{L} \in \text{Theories}
\end{align*}
\]

Denote the objects of $\mathcal{L}$ by $T^A$ with $A$ a finite set.

Let $J_L$ be the least Grothendieck topology on $\mathcal{L}$ such that $T^1$ is covered by the sieve generated by the family

\[
(f : T^A \to T^1 | A \in \mathsf{fSet}, f \in \ell(\#A))
\]

Example (For $L(X) = 1 + 2X^3$)
A coverage on a free algebraic theory

\[
\begin{array}{ccc}
\text{Set}^\mathbb{N} \xrightarrow{\text{free}} \text{Theories} \\
\text{forget} \downarrow \\
\end{array}
\]

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**Example (For \( L(X) = 1 + 2X^3 \))**

\[
\begin{array}{cccc}
T^0 & T^3 & T^3 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
T^1 & T^3 & T^3 & T^3 \\
\end{array}
\]
The canonical $L(T) \rightarrow T$

$$L(X) \in \mathbb{N}[X] \rightarrow \ell \in \text{Set}^\mathbb{N}_{\text{free}} \rightarrow \mathcal{L} \in \text{Theories}$$

The basic cover

$$(f : T^A \rightarrow T^1 \mid A \in \text{fSet}, f \in \ell(\#A)) \quad \text{in } \mathcal{L}$$

induces a map
The canonical $L(T) \to T$

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$$L(T) \to T \text{ in } \mathbf{Sh}(\mathcal{L}, J_L).$$

Example (With $L(X) = 1 + 2X^3$)
The canonical $L(T) \to T$

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induces a map

$L(T) \to T \text{ in } \text{Sh}(\mathcal{L}, J_\mathcal{L})$.

Example (With $L(X) = 1 + 2X^3$)

\[
\begin{array}{ccc}
1 & T^3 & T^3 \\
\downarrow & \downarrow & \downarrow \\
T & T^3 & T^3
\end{array}
\]

in $\mathcal{L}$

$1 + T^3 + T^3 = L(T)$

in $\text{Sh}(\mathcal{L}, J_\mathcal{L})$
Basic operations are mono and disjoint

\[ L(X) \in \mathbb{N}[X] \quad \rightarrow \quad \ell \in \text{Set}^{\mathbb{N}} \quad \overset{\text{free}}{\rightarrow} \quad \mathcal{L} \in \text{Theories} \]

The basic cover \((f : T^A \rightarrow T^1 \mid A \in \text{fSet}, f \in \ell(\#A))\) in \(\mathcal{L}\) induces a map \(L(T) \rightarrow T\) in \(\text{Sh}(\mathcal{L}, J_\mathcal{L})\).

**Lemma**
Basic operations are mono and disjoint

\[ L(X) \in \mathbb{N}[X] \rightarrow \ell \in \text{Set}^\mathbb{N} \rightarrow \mathcal{L} \in \text{Theories} \]

The basic cover \((f : T^A \rightarrow T^1 \mid A \in \text{fSet}, f \in \ell(\#A))\) in \(\mathcal{L}\) induces a map \(L(T) \rightarrow T\) in \(\text{Sh}(\mathcal{L}, J_L)\).

**Lemma**

The canonical map \(L(T) \rightarrow T\) in \(\text{Sh}(\mathcal{L}, J_L)\) is an iso.

**Proof.**

It is epic because it is induced by a covering family.
Basic operations are mono and disjoint

\[ L(X) \in \mathbb{N}[X] \xrightarrow{-} \ell \in \text{Set}^{\mathbb{N}} \xrightarrow{\text{free}} \mathcal{L} \in \text{Theories} \]

The basic cover \((f : T^A \to T^1 \mid A \in \mathbf{fSet}, f \in \ell(\#A))\) in \(\mathcal{L}\) induces a map \(L(T) \to T\) in \(\text{Sh}(\mathcal{L}, J_L)\).

**Lemma**

The canonical map \(L(T) \to T\) in \(\text{Sh}(\mathcal{L}, J_L)\) is an iso.

**Proof.**

It is epic because it is induced by a covering family.
It is monic because maps in \((f : T^A \to T \mid A \in \mathbf{fSet}, f \in \ell(\#A))\) are monic and ‘disjoint’ (in \(\mathcal{L}\)).

\[ \text{So we have } \mathbb{N}[X]/(X = L(X)) \to \mathcal{B}(\text{Sh}(\mathcal{L}, J_L)). \]
The canonical surjection

So we have $\mathbb{N}[X]/(X = L(X)) \to \mathcal{B}(\text{Sh}(\mathcal{L}, J_L))$. Moreover:

\[
\begin{array}{ccc}
\mathbb{N}[X] & \rightarrow & \mathbb{N}[X]/(X = L(X)) \\
\downarrow & & \downarrow \\
\mathcal{B}(\text{Fam}\mathcal{L}) & \rightarrow & \mathcal{B}(\text{Fam}(\mathcal{L}, J_L)) \rightarrow \mathcal{B}(\text{Sh}(\mathcal{L}, J_L))
\end{array}
\]

so

$\mathbb{N}[X]/(X = L(X)) \rightarrow \mathcal{B}(\text{Fam}(\mathcal{L}, J_L))$

is surjective.

To prove that it is an iso, it is enough to build a retraction.
The not-so-easy part (sketch)

\[ \mathcal{B}(\text{Fam}(\mathcal{L}, J_{\mathcal{L}})) \longrightarrow \mathbb{N}[X]/(X = L(X)) \]
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\[ \mathcal{B}(\text{Fam}(\mathcal{L}, J_L)) \to \mathbb{N}[X]/(X = L(X)) \]

Use that \( J_L \) is induced by a ‘good’ basis \( K \).
The not-so-easy part (sketch)

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Use that \( J_L \) is induced by a ‘good’ basis \( K \).

\[ \mathcal{L} \longrightarrow \text{Fam}\mathcal{L} \longrightarrow \text{Fam}(\mathcal{L}, J_L) \]
The not-so-easy part (sketch)

\[ \mathcal{B}(\text{Fam}(\mathcal{L}, J_L)) \rightarrow \mathbb{N}[X]/(X = L(X)) \]

Use that \( J_L \) is induced by a ‘good’ basis \( K \).

\[ \mathcal{L} \rightarrow \text{Fam}\mathcal{L} \rightarrow \text{Fam}(\mathcal{L}, J_L) \]

\[ \mathcal{B}\mathcal{L} \rightarrow \mathcal{B}(\text{Fam}\mathcal{L}) \rightarrow \mathcal{B}(\text{Fam}(\mathcal{L}, J_L)) \]

‘compatible’ with \( K \)

\[ \rightarrow \mathbb{N}[X]/(X = L(X)) \]
The not-so-easy part

\[ \mathcal{L} \rightarrow \text{Fam}\mathcal{L} \rightarrow \text{Fam}(\mathcal{L}, J_L) \]

\[ [T^n] \]

\[ \mathcal{B}\mathcal{L} \rightarrow \mathcal{B}(\text{Fam}\mathcal{L}) \rightarrow \mathcal{B}(\text{Fam}(\mathcal{L}, J_L)) \]

\[ (\mathbb{N}, +) \xrightarrow{\text{compatible with } K} \mathbb{N}[X]/(X = L(X)) \]

\[ n \rightarrow [X^n] \]

END OF SKETCH.
A family \((f_i : C_i \to C \mid i \in I)\) in a category \(\mathcal{C}\) is called monic if the existence of a commutative diagram

\[
\begin{array}{ccc}
\cdot & \xrightarrow{h} & C_j \\
g \downarrow & & \downarrow f_j \\
C_i & \xrightarrow{f_i} & C
\end{array}
\]

implies \(i = j\) and \(g = h\).
A family \((f_i : C_i \to C \mid i \in I)\) in a category \(\mathcal{C}\) is called **monic** if the existence of a commutative diagram

\[
\begin{array}{ccc}
\cdots & \xrightarrow{h} & C_j \\
\downarrow{g} & & \downarrow{f_j} \\
C_i & \xrightarrow{f_i} & C
\end{array}
\]

implies \(i = j\) and \(g = h\).

**Lemma**

A finite family \((f_i : C_i \to C \mid i \in I)\) is monic if and only if \([f_i \mid i \in I] : \sum_{i \in I} C_i \to C\) is monic in \(\text{Fam}\mathcal{C}\).
A family \((f_i : C_i \to C \mid i \in I)\) in a category \(C\) is called monic if the existence of a commutative diagram

\[
\begin{array}{ccc}
C_i & \xrightarrow{f_i} & C \\
\downarrow g & & \downarrow f_j \\
\cdot & \xrightarrow{h} & C_j \\
\downarrow & & \downarrow f_j \\
C_j & & C
\end{array}
\]

implies \(i = j\) and \(g = h\).

**Lemma**

A finite family \((f_i : C_i \to C \mid i \in I)\) is monic if and only if \([f_i \mid i \in I] : \sum_{i \in I} C_i \to C\) is monic in Fam\(C\).

A family is proper if it is finite and monic.
Let $C$ be a small category with finite products.
Compatibility for finite families

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Let $R$ be a rig.
Compatibility for finite families

Let $C$ be a small category with finite products.

Let $R$ be a rig.

Let $\gamma : \mathcal{B}C \to R$ be a morphism of multiplicative monoids.
Compatibility for finite families

Let $\mathcal{C}$ be a small category with finite products.

Let $R$ be a rig.

Let $\gamma : \mathcal{B}C \to R$ be a morphism of multiplicative monoids.

**Definition**

$\gamma$ is **compatible** with a finite family $(f_i : C_i \to C \mid i \in I)$ in $\mathcal{C}$, if

$$\gamma[C] = \sum_{i \in I} \gamma[C_i].$$
Let $\mathcal{C}$ be a small category with finite products.

Let $R$ be a rig.

Let $\gamma : \mathcal{B} \mathcal{C} \to R$ be a morphism of multiplicative monoids.

**Definition**

$\gamma$ is compatible with a finite family $(f_i : C_i \to C \mid i \in I)$ in $\mathcal{C}$, if $\gamma[C] = \sum_{i \in I} \gamma[C_i]$.

Following Blass, we could call $\sum_{i \in I} \gamma[C_i]$ the weight of the family $(f_i : C_i \to C \mid i \in I)$.

Example) ⇝
Example, with $L(X) = 1 + X^2$

$$
\begin{align*}
\mathfrak{B}L & \overset{\gamma}{\longrightarrow} \mathbb{N}[X]/(X = 1 + X^2) \\
[T] & \longrightarrow X
\end{align*}
$$

is compatible with:
Example, with $L(X) = 1 + X^2$

$$\mathcal{B} \mathcal{L} \xrightarrow{\gamma} \mathbb{N}[X]/(X = 1 + X^2)$$

$[T] \quad \xrightarrow[]{} \quad X$

is compatible with:

$$\begin{array}{c}
1 \\
\downarrow \\
T
\end{array} \quad \begin{array}{c}
T^2 \\
\downarrow \\
T
\end{array}$$

because

$$\gamma[1] + \gamma[T^2] = 1 + X^2 = X = \gamma[T]$$

**Definition**

A multiplicative $\gamma : \mathcal{B}(\mathcal{L}) \to R$ is **compatible** with a basis $K$ on $\mathcal{L}$ if $\gamma$ is compatible with every $K$-cover.
Let $C$ be a small category with finite products.

Let $R$ be a rig.

Let $\gamma : \mathbb{B}C \to R$ be a map of multiplicative monoids.

**Proposition**

If $K$ is a 'good' basis on $C$ and $\gamma$ is compatible with it then there exists a unique $\gamma' : \mathbb{B}(\text{Fam}(C, K)) \to R$ of rigs such that $\mathbb{B}(\text{Fam}(C)) \to \mathbb{B}(\text{Fam}(C, K)) \gamma' \to R$

In the intended application, $C = L$ and $R = \mathbb{N}[[X]] / (X = L(X))$ but...

Does $JL$ have a 'good' basis?
Let $\mathcal{C}$ be a small category with finite products.

Let $R$ be a rig.

Let $\gamma : \mathcal{B}\mathcal{C} \to R$ be a map of multiplicative monoids.

**Proposition**

If $K$ is a ‘good’ basis on $\mathcal{C}$ and $\gamma$ is compatible with it then there exists a unique $\gamma' : \mathcal{B}(\text{Fam}(\mathcal{C}, K)) \to R$ of rigs such that

$$\mathcal{B}(\text{Fam}\mathcal{C}) \to \mathcal{B}(\text{Fam}(\mathcal{C}, K))$$

$$\downarrow \gamma'$$

$$R$$
Let $C$ be a small category with finite products.

Let $R$ be a rig.

Let $\gamma : \mathcal{B}C \to R$ be a map of multiplicative monoids.

**Proposition**

If $K$ is a ‘good’ basis on $C$ and $\gamma$ is compatible with it then there exists a unique $\gamma' : \mathcal{B}(\text{Fam}(C, K)) \to R$ of rigs such that

$$\mathcal{B}(\text{Fam}C) \longrightarrow \mathcal{B}(\text{Fam}(C, K)) \xrightarrow{\gamma'} R$$

In the intended application, $C = \mathcal{L}$ and $R = \mathbb{N}[X]/(X = L(X))$ but...
Let $\mathcal{C}$ be a small category with finite products.

Let $R$ be a rig.

Let $\gamma : \mathcal{BC} \to R$ be a map of multiplicative monoids.

**Proposition**

If $K$ is a ‘good’ basis on $\mathcal{C}$ and $\gamma$ is compatible with it then there exists a unique $\gamma' : \mathcal{B}(\text{Fam}(\mathcal{C}, K)) \to R$ of rigs such that

\[
\begin{array}{ccc}
\mathcal{B}(\text{FamC}) & \longrightarrow & \mathcal{B}(\text{Fam}(\mathcal{C}, K)) \\
\downarrow & & \downarrow \\
R & & R
\end{array}
\]

In the intended application, $\mathcal{C} = \mathcal{L}$ and $R = \mathbb{N}[X]/(X = L(X))$

but... Does $J_L$ have a ‘good’ basis?
Blass’ Lemma

For $L(X) = 1 + X^2$. 
Blass’ Lemma

For $L(X) = 1 + X^2$.

Lemma

The following four Grothendieck topologies on $\mathcal{L}$ coincide:

1. The smallest topology for which $T$ is covered by the cospan $1 \to T \leftarrow T^2$.

2. The smallest topology in which, for each $n$, each $T^k$ is covered by the set $S_n(T^k)$.

3. The topology where the covering sieves of any $T^k$ are those sieves that include $S_n(T^k)$ for some $n$.

4. The topology where the covering sieves of any $T^k$ are those sieves that include a finite family of maps $T^{k_i} \to T^k$ such that every map $1 \to T^k$ factors through a map from the finite family.

(It is part of the assertion of the lemma that the collections of sieves described in (3) and (4) are topologies.)
Blass’ Lemma

For $L(X) = 1 + X^2$.

**Lemma**

The following four Grothendieck topologies on $\mathcal{L}$ coincide:

1. The smallest topology for which $T$ is covered by the cospan $1 \to T \leftarrow T^2$.

2. .

3. .

4. The topology where the covering sieves of any $T^k$ are those sieves that include a finite family of maps $T^{k_i} \to T^k$ such that every map $1 \to T^k$ factors through a map from the finite family.

(...)


Examples of ‘Blass-covers’

\[ id : T^A \rightarrow T^A, \]
Examples of ‘Blass-covers’

$id : T^A \to T^A$, the cover of basic ops,
Examples of ‘Blass-covers’

$id : T^A \rightarrow T^A$, the cover of basic ops,

\[ T = 1 \times T \]
Examples of ‘Blass-covers’

$id : T^A \to T^A$, the cover of basic ops,

\[
T = 1 \times T
\]

\[
T^2 = T \times T
\]

\[
T^2 \times T
\]
Normed categories [Lawevere’s ‘Metric spaces’ paper]

Let $\mathcal{V}, \otimes, \mathbb{K}$ be a monoidal category.

**Definition**

A $\mathcal{V}$-normed category is a category $\mathcal{C}$ together with the assignment of an object $d_{X,Y}f = df$ of $\mathcal{V}$ to every map $f : X \to Y$ in $\mathcal{C}$,
Normed categories [Lawevere’s ‘Metric spaces’ paper]

Let \((\mathcal{V}, \otimes, k)\) be a monoidal category.

**Definition**

A \(\mathcal{V}\)-normed category is a category \(\mathcal{C}\) together with the assignment of an object \(d_{X,Y} f = d_f\) of \(\mathcal{V}\) to every map \(f : X \to Y\) in \(\mathcal{C}\), the assignment of a morphism

\[
(d g) \otimes (d f) \to d(g f)
\]

in \(\mathcal{V}\) for every pair of maps \(f : X \to Y\) and \(g : Y \to Z\) in \(\mathcal{C}\), and
Normed categories [Lawvere’s ‘Metric spaces’ paper]

Let $(\mathcal{V}, \otimes, k)$ be a monoidal category.

**Definition**

A **\(\mathcal{V}\)-normed category** is a category \(\mathcal{C}\) together with the assignment of an object \(d_{X,Y}f = d_f\) of \(\mathcal{V}\) to every map \(f : X \to Y\) in \(\mathcal{C}\), the assignment of a morphism

\[
(\mathcal{d}g) \otimes (\mathcal{d}f) \to \mathcal{d}(gf)
\]

in \(\mathcal{V}\) for every pair of maps \(f : X \to Y\) and \(g : Y \to Z\) in \(\mathcal{C}\), and the assignment of a morphism

\[
k \to \mathcal{d}id_X
\]

in \(\mathcal{V}\) for every object \(X\) in \(\mathcal{C}\), subject to
Let \((\mathcal{V}, \otimes, k)\) be a monoidal category.

**Definition**

A \(\mathcal{V}\)-normed category is a category \(C\) together with the assignment of an object \(d_{X,Y}f = df\) of \(\mathcal{V}\) to every map \(f : X \to Y\) in \(C\), the assignment of a morphism 

\[
(dg) \otimes (df) \to d(gf)
\]

in \(\mathcal{V}\) for every pair of maps \(f : X \to Y\) and \(g : Y \to Z\) in \(C\), and the assignment of a morphism 

\[
k \to d(id_X)
\]

in \(\mathcal{V}\) for every object \(X\) in \(C\), subject to the evident associativity and unit conditions (that we need not emphasize because they automatically hold in our main example of base monoidal category).
The monoidal category $\mathbb{N}_\infty$

Let $(\mathbb{N}, +, 0)$ be the usual commutative monoid of natural numbers under addition and consider its extension $\mathbb{N}_\infty = (\mathbb{N} + \{\infty\}, +, 0)$ with an element $\infty$ such that, for every $n \in \mathbb{N} + \{\infty\}$, 

$$\infty + n = \infty = \infty + n.$$
The monoidal category $\mathbb{N}_\infty$

Let $(\mathbb{N}, +, 0)$ be the usual commutative monoid of natural numbers under addition and consider its extension $\mathbb{N}_\infty = (\mathbb{N} + \{\infty\}, +, 0)$ with an element $\infty$ such that, for every $n \in \mathbb{N} + \{\infty\}$, 
\[\infty + n = \infty = \infty + n.\]

The monoid structure induces a total order $(\mathbb{N}_\infty, \leq)$ with $\infty$ as terminal object. Moreover, addition extends to a symmetric monoidal structure on the category $(\mathbb{N}_\infty, \leq)$. The resulting monoidal category $((\mathbb{N}_\infty, \leq), +, 0)$ will be denoted simply by $(\mathbb{N}_\infty, \leq)$. 
Categories normed in \((\mathbb{N}_\infty, \leq)\)

In concrete terms, an \((\mathbb{N}_\infty, \leq)\)-normed category is a category \(\mathcal{C}\) equipped with a collection \((\varrho_{X,Y} : \mathcal{C}(X, Y) \to \mathbb{N}_\infty \mid X, Y \in \mathcal{C}\)) of functions such that:

\[(\varrho g) + (\varrho f) \leq \varrho(gf)\]

holds for every \(f : X \to Y\) and \(g : Y \to Z\).

(It automatically holds that

\[0 \leq \varrho(id_X)\]

for every \(X\) in \(\mathcal{C}\).)
Categories normed in \((\mathbb{N}_\infty, \leq)\)

In concrete terms, an \((\mathbb{N}_\infty, \leq)\)-normed category is a category \(\mathcal{C}\) equipped with a collection \((d_{X,Y} : \mathcal{C}(X,Y) \to \mathbb{N}_\infty \mid X,Y \in \mathcal{C})\) of functions such that:

\[(d_g) + (d_f) \leq d(gf)\]

holds for every \(f : X \to Y\) and \(g : Y \to Z\).

(It automatically holds that

\[0 \leq d(id_X)\]

for every \(X\) in \(\mathcal{C}\).)

(If the \(\mathcal{C}\) has a terminal object then we may entertain the condition that \(d_p = \infty\) for every point \(p : 1 \to \mathcal{C}\).)
Examples of normed categories

Let $L(X) \in \mathbb{N}[X]$ and let $\mathcal{L}$ be the resulting free theory.

**Proposition**

The assignment $\varnothing$ that sends

1. each basic constant $1 \to T$ in $\mathcal{L}$ to $\infty \in \mathbb{N}_\infty$ and
2. each basic non-constant $T^n \to T$ to $1 \in \mathbb{N} \subseteq \mathbb{N}_\infty$

extends to a norm making $\mathcal{L}$ into a $(\mathbb{N}_\infty, \leq)$-normed category.
Examples of normed categories

Let \( L(X) \in \mathbb{N}[X] \) and let \( \mathcal{L} \) be the resulting free theory.

**Proposition**

The assignment \( \partial \) that sends

1. each basic constant \( 1 \rightarrow T \) in \( \mathcal{L} \) to \( \infty \in \mathbb{N}_\infty \) and
2. each basic non-constant \( T^n \rightarrow T \) to \( 1 \in \mathbb{N} \subseteq \mathbb{N}_\infty \)

extends to a norm making \( \mathcal{L} \) into a \((\mathbb{N}_\infty, \leq)\)-normed category.

Intuition: if you think of a map \( f : T^A \rightarrow T \) in \( \mathcal{L} \) as a tree with leaves in \( A \) then \( \partial f \) is the length of a shortest path to a leaf.
Example with $L(X) = 1 + X^2$

Constant $c : 1 \to T$, binary operation $b : T^2 \to T$. 
Example with $L(X) = 1 + X^2$

Constant $c : 1 \to T$, binary operation $b : T^2 \to T$.

$$1 \times T^2 \xrightarrow{c \times id} T \times T^2 \quad T \times T^2 \xrightarrow{id \times b} T \times T = T^2$$

$\infty \land 0 = 0$  
$0 \land 1 = 0$
Example with $L(X) = 1 + X^2$

Constant $c : 1 \to T$, binary operation $b : T^2 \to T$.

\[
\begin{align*}
1 \times T^2 &\xrightarrow{c \times id} T \times T^2 & T \times T^2 &\xrightarrow{id \times b} T \times T = T^2 \\
\infty \land 0 &= 0 & 0 \land 1 &= 0
\end{align*}
\]

\[
\begin{align*}
T^2 &= 1 \times T^2 \xrightarrow{c \times b} T \times T = T^2 \\
\infty \land 1 &= 1
\end{align*}
\]
Example with \( L(X) = 1 + X^2 \)

Constant \( c : 1 \rightarrow T \), binary operation \( b : T^2 \rightarrow T \).

\[
1 \times T^2 \xrightarrow{c \times id} T \times T^2 \quad \quad T \times T^2 \xrightarrow{id \times b} T \times T = T^2
\]

\[
\infty \land 0 = 0 \quad \quad 0 \land 1 = 0
\]

\[
T^2 = 1 \times T^2 \xrightarrow{c \times b} T \times T = T^2
\]

\[
\infty \land 1 = 1
\]

\[
\partial (c \times id) + \partial (id \times b) = 0 + 0 = 0 \leq
\]
Example with $L(X) = 1 + X^2$

Constant $c : 1 \to T$, binary operation $b : T^2 \to T$.

\[
1 \times T^2 \xrightarrow{c \times id} T \times T^2 \quad \quad T \times T^2 \xrightarrow{id \times b} T \times T = T^2
\]

\[
\infty \land 0 = 0 \quad \quad 0 \land 1 = 0
\]

\[
T^2 = 1 \times T^2 \xrightarrow{c \times b} T \times T = T^2 \quad \quad \infty \land 1 = 1
\]

\[
\delta(c \times id) + \delta(id \times b) = 0 + 0 = 0 \leq 1 = \delta(c \times b)
\]
Let $\mathcal{C}$ be an $(\mathbb{N}_\infty, \leq)$-normed category with ‘norm’ $\varnothing$.
Let $F$ be a family of maps in $\mathcal{C}$ with codomain $Y$.

Is $\mathcal{K}_Y$ the basis for a Grothendieck topology?
Ample families

Let $\mathcal{C}$ be an $(\mathbb{N}_\infty, \leq)$-normed category with ‘norm’ $\vartheta$.

Let $F$ be a family of maps in $\mathcal{C}$ with codomain $Y$.

**Definition**

For $k \in \mathbb{N}$, the family $F$ is called $k$-ample if for every $f : X \to Y$ in $\mathcal{C}$, $\vartheta f \geq k$ implies that $f$ factors through some map in $F$. 

Is $K_Y$ the basis for a Grothendieck topology?
Ample families

Let $\mathcal{C}$ be an $(\mathbb{N}_\infty, \leq)$-normed category with ‘norm’ $\partial$.

Let $F$ be a family of maps in $\mathcal{C}$ with codomain $Y$.

**Definition**

For $k \in \mathbb{N}$, the family $F$ is called *$k$-ample* if for every $f : X \to Y$ in $\mathcal{C}$, $\partial f \geq k$ implies that $f$ factors through some map in $F$.

The family $F$ is called *ample* if it is $k$-ample for some $k \in \mathbb{N}$. 
Ample families

Let $C$ be an $(\mathbb{N}_\infty, \leq)$-normed category with ‘norm’ $\mathcal{d}$.

Let $F$ be a family of maps in $C$ with codomain $Y$.

**Definition**

For $k \in \mathbb{N}$, the family $F$ is called $k$-ample if for every $f : X \to Y$ in $C$, $\mathcal{d}f \geq k$ implies that $f$ factors through some map in $F$.

The family $F$ is called ample if it is $k$-ample for some $k \in \mathbb{N}$.

(Notice: if $\mathcal{d}p = \infty$ for every point $p$, then ample families contain all points.)
Ample families

Let $\mathcal{C}$ be an $(\mathbb{N}_\infty, \leq)$-normed category with ‘norm’ $\partial$.
Let $F$ be a family of maps in $\mathcal{C}$ with codomain $Y$.

**Definition**

For $k \in \mathbb{N}$, the family $F$ is called $k$-ample if for every $f : X \to Y$ in $\mathcal{C}$, $\partial f \geq k$ implies that $f$ factors through some map in $F$.
The family $F$ is called **ample** if it is $k$-ample for some $k \in \mathbb{N}$.

(Notice: if $\partial p = \infty$ for every point $p$, then ample families contain all points.)

Let $\mathcal{A}_Y$ be the set of ample and proper families with codomain $Y$.
Is $\mathcal{A}$ the basis for a Grothendieck topology?
A generalization of Blass’ Lemma

Let $L(X) \in \mathbb{N}[X]$. Let $(\mathcal{L}, \vartheta)$ be the resulting free theory normed in $\mathbb{N}_\infty$. Let $\mathcal{K}(T^A)$ be the set of ample and proper families on $T^A$.

Lemma
A generalization of Blass’ Lemma

Let $L(X) \in \mathbb{N}[X]$. Let $(\mathcal{L}, \mathfrak{d})$ be the resulting free theory normed in $\mathbb{N}_\infty$. Let $\mathcal{K}(T^A)$ be the set of ample and proper families on $T^A$.

Lemma

If $L(X)$ is non-constant then $\mathcal{K}$ is the basis of a Grothendieck topology and moreover,
A generalization of Blass’ Lemma

Let $L(X) \in \mathbb{N}[X]$.
Let $(L, \mathcal{O})$ be the resulting free theory normed in $\mathbb{N}_\infty$.
Let $\mathcal{K}(T^A)$ be the set of ample and proper families on $T^A$.

Lemma

If $L(X)$ is non-constant then $\mathcal{K}$ is the basis of a Grothendieck topology and moreover, the following topologies on $L$ coincide:

1. $J_L$ (i.e. the smallest topology containing the basic cover).
2. -
3. -
4. The topology generated by the basis $\mathcal{K}$. 
Strictly Ample families

Definition

A family $F$ on $Y$ is strictly $n$-ample if it is $n$-ample and for every $f \in F$, $\delta f \geq n$. 

Proper and strictly $n$-ample families are unique up to iso in the following sense.

Lemma

If $F = (f_i : X_i \to Y | i \in I)$ and $H = (h_j : A_j \to Y | j \in J)$ are both strictly $n$-ample and proper families then there exists a unique bijection $\phi : I \to J$ such that for every $i \in I$, $h_{\phi i}$ is iso to $f_i$ over $Y$. 
Strictly Ample families

Definition

A family $F$ on $Y$ is strictly $n$-ample if it is $n$-ample and for every $f \in F$, $\partial f \geq n$. A family is strictly ample if it is strictly $n$-ample for some $n$. 
Strictly Ample families

Definition

A family $F$ on $Y$ is strictly $n$-ample if it is $n$-ample and for every $f \in F$, $\delta f \geq n$. A family is strictly ample if it is strictly $n$-ample for some $n$.

Proper and strictly $n$-ample families are unique up to iso in the following sense.

Lemma

If $F = (f_i : X_i \to Y \mid i \in I)$ and $H = (h_j : A_j \to Y \mid j \in J)$ are both strictly $n$-ample and proper families then there exists a unique bijection $\phi : I \to J$ such that for every $i \in I$, $h_{\phi i}$ is iso to $f_i$ over $Y$. 
Blass’ Lemma again

Let $L(X) \in \mathbb{N}[X]$.

Let $(\mathcal{L}, \mathcal{O})$ be the resulting free theory normed in $\mathbb{N}_\infty$.

Let $\mathcal{K}(T^A)$ be the set of ample and proper families on $T^A$.

**Lemma**

If $L(X)$ is non-constant then $\mathcal{K}$ is the basis of a Grothendieck topology and moreover, the following topologies on $\mathcal{L}$ coincide:

1. $J_L$ (i.e. the smallest topology containing the basic cover).
2. The smallest topology for which $T^A$ is covered by the strictly ample and proper families with codomain $T^A$.
3. The topology where the covering sieves of $T^A$ are those that include some strictly ample and proper family.
4. The topology generated by the basis $\mathcal{K}$. 
The main result

Theorem

For any \( L(X) \in \mathbb{N}[X] \), the canonical
\[ \mathbb{N}[X]/(X = L(X)) \to \mathcal{B}(\text{Fam}(\mathcal{L}, J_L)) \] is an isomorphism.

Proof.

For non-constant \( L(X) \in \mathbb{N}[X] \), \( \text{Fam}(\mathcal{L}, J_L) \cong \text{Fam}(\mathcal{L}, \mathcal{K}) \) by the ‘ample’ Blass Lemma.
The main result

Theorem

For any \(L(X) \in \mathbb{N}[X]\), the canonical 
\[\mathbb{N}[X]/(X = L(X)) \to \mathcal{B}(\text{Fam}(\mathcal{L}, J_L))\] is an isomorphism.

Proof.

For non-constant \(L(X) \in \mathbb{N}[X]\), \(\text{Fam}(\mathcal{L}, J_L) \cong \text{Fam}(\mathcal{L}, \mathcal{K})\) by the ‘ample’ Blass Lemma.
An ‘ample’ variant of Blass’ argument shows that the canonical 
\[\mathcal{B} \mathcal{L} \xrightarrow{\cong} \mathbb{N} \xrightarrow{\_} \mathbb{N}[X]/(X = L(X))\]
is compatible with \(\mathcal{K}\),
The main result

**Theorem**

For any \( L(X) \in \mathbb{N}[X] \), the canonical
\[ \mathbb{N}[X]/(X = L(X)) \rightarrow \mathcal{B}(\text{Fam}(\mathcal{L}, J_L)) \] is an isomorphism.

**Proof.**

For non-constant \( L(X) \in \mathbb{N}[X] \), \( \text{Fam}(\mathcal{L}, J_L) \cong \text{Fam}(\mathcal{L}, \mathcal{K}) \) by the ‘ample’ Blass Lemma.

An ‘ample’ variant of Blass’ argument shows that the canonical
\[ \mathcal{B} \mathcal{L} \xrightarrow{\cong} \mathbb{N} \xrightarrow{} \mathbb{N}[X]/(X = L(X)) \]
is compatible with \( \mathcal{K} \), so it extends ‘along \( \mathcal{L} \rightarrow \text{Fam}(\mathcal{L}, \mathcal{K}) \)’ to a
\[ \mathcal{B}(\text{Fam}(\mathcal{L}, \mathcal{K})) \rightarrow \mathbb{N}[X]/(X = L(X)) \]
that is a retraction to \( \mathbb{N}[X]/(X = L(X)) \rightarrow \mathcal{B}(\text{Fam}(\mathcal{L}, J_L)) \).
\[ \Box \]
Developments and compatibility
Pullbacks of the basic family

Let $\mathcal{T}$ be an algebraic theory.

Let $B$ be a family of maps in $\mathcal{T}$ with codomain $\mathcal{T}$. 
Let $T$ be an algebraic theory.

Let $B$ be a family of maps in $T$ with codomain $T$.

For any finite $I$ and $i \in I$, let $\pi_i^*F$ the pullback family

$$
\begin{array}{c}
\pi_i^*F \\
\downarrow \\
T^I \\
\downarrow \\
T
\end{array}
$$

Families of the form $\pi_i^*F$ will be called primitive.
Example with $L(X) = 1 + X^2$

Let $\mathcal{L}$ be the free theory on $c : 1 \to T$ and $b : T^2 \to T$. 
Example with $L(X) = 1 + X^2$

Let $\mathcal{L}$ be the free theory on $c : 1 \to T$ and $b : T^2 \to T$. Let $F = \{c, b\}$ the family on the right below.
Example with $L(X) = 1 + X^2$

Let $L$ be the free theory on $c : 1 \to T$ and $b : T^2 \to T$. Let $F = \{c, b\}$ the family on the right below. So that $\pi_0^* F$ is the family on the left below.
Example with $L(X) = 1 + X^2$

Let $\mathcal{L}$ be the free theory on $c : 1 \to T$ and $b : T^2 \to T$. Let $F = \{c, b\}$ the family on the right below. So that $\pi_0^* F$ is the family on the left below.

![Diagram](image)

Similarly, $\pi_1^* F$ is

![Diagram](image)
Developments

\[ T = 1 \times T \]

\[ T^2 \]

\[ T^2 \times T \]

\[ \pi_0^* F \]

\[ c \times T \]

\[ b \times T \]

\[ F \]

\[ b \]

\[ c \]

\[ 1 \]

\[ 1 \]

\[ T \]

\[ T \times T \]
Developments

\[ T = 1 \times T \]

\[ F \]

\[ c \]

\[ T^2 \]

\[ b \]

\[ \pi_0^* F \]

\[ c \times T \]

\[ b \times T \]

\[ T \times T \]

\[ T^2 \times T \]

\[ 1 \]

\[ c \]

\[ b \]

\[ T \]

\[ 1 \]

\[ T \]

\[ 1 \]

\[ T^2 \]

\[ T^3 \]

\[ 1 \]

\[ T \]

\[ c \]

\[ b(c, c) \]

\[ b(c, b(x, y)) \]

\[ b(b(x, y), z) \]
Slogan/Theorem

Developments make sense in any algebraic theory equipped with a distinguished ‘cover’.
Let $\gamma : \mathcal{B} \mathcal{L} \to R = \mathbb{N}[X]/(X = 1 + X^2)$ be the morphism of multiplicative monoids sending $[T]$ to $X$. 
Compatibility for bases (again)

Let $\gamma : \mathcal{BL} \rightarrow R = \mathbb{N}[X]/(X = 1 + X^2)$ be the morphism of multiplicative monoids senting $[T]$ to $X$.

Recall!! $\gamma$ is compatible with $(f_i : C_i \rightarrow C \mid i \in I)$, $\gamma[C] = \sum_{i \in I} \gamma[C_i]$. 
Compatibility for bases (again)

Let \( \gamma : \mathcal{B} \mathcal{L} \to R = \mathbb{N}[X]/(X = 1 + X^2) \) be the morphism of multiplicative monoids sending \([T]\) to \(X\).

Recall!! \( \gamma \) is compatible with \( (f_i : C_i \to C \mid i \in I) \),
\[
\gamma[C] = \sum_{i \in I} \gamma[C_i].
\]

\[
2 + X^2 + X^3 = 1 + (1 + X^2) + X^3 = 1 + X + X^3 = 1 + (1 + X^2)X = 1 + X^2 = X
\]
Compatibility for bases (again)

Let $\gamma : \mathcal{B} \rightarrow R = \mathbb{N}[X]/(X = 1 + X^2)$ be the morphism of multiplicative monoids sending $[T]$ to $X$.

Recall!! $\gamma$ is compatible with $(f_i : C_i \rightarrow C \mid i \in I)$, $\gamma[C] = \sum_{i \in I} \gamma[C_i]$.

\[
2 + X^2 + X^3 = 1 + (1 + X^2) + X^3 = 1 + X + X^3 = 1 + (1 + X^2)X = 1 + X^2 = X
\]

Slogan/Theorem) If $\gamma$ is compatible with $F$ then it is compatible with any development of $F$. 
Developments to level 2

\[ T = 1 \times T \]

\[ T^2 \]

\[ T^2 \times 1 \]

\[ T^2 \times T \]

\[ T^2 \times T^2 \]

\[ T \times T \]

\[ F \]

\[ F \]

\[ c \rightarrow b \]

\[ c \times T \]

\[ \pi_0^* F \]

\[ \pi_1^* F \]

\[ T^2 \times c \]

\[ T^2 \times b \]

\[ b \times T \]

\[ (c, b) \]

\[ (x, y) \]

\[ (x', y') \]
Developments to level 2

\[ (\pi_0^* F \times 1) \rightarrow (c \times T) \rightarrow T = 1 \times T \]

\[ (\pi_1^* F \times 1) \rightarrow (b \times T) \rightarrow T = 1 \times T \]

\[ (\pi_0^* F \times T) \rightarrow (c \times T) \rightarrow T \times T \]

\[ (\pi_1^* F \times T) \rightarrow (b \times T) \rightarrow T \times T \]

\[ T \rightarrow S_2(T) \rightarrow T^2 \rightarrow T^3 \rightarrow T^4 \]

1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1

\[ c \quad b(c, c) \quad b(c, b(x, y)) \quad b(b(x, y), c) \quad b(b(x, y), b(x', y')) \]
Further comments
The classifying role of $\text{Sh}({\mathcal{L}}, J_L)$

**Theorem**

*If $\mathcal{T}$ is the free algebraic theory generated by $\ell \in \text{Set}^\mathbb{N}$. Then $\hat{\mathcal{T}}$ classifies the theory presented by the same operations and*

1. *(Basic ops are injective)* For every $n \in \mathbb{N}$ and $f \in \ell n$:

   $$f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n) \vdash_{x_1, \ldots, x_n, y_1, \ldots, y_n} \bigwedge_{i=1}^{n} x_i = y_i$$

2. *(Basic ops are disjoint)* For all $f \in \ell m$, $f' \in \ell n$ and $f \neq f'$:

   $$f(x_1, \ldots, x_m) = f'(y_1, \ldots, y_n) \vdash_{x_1, \ldots, x_m, y_1, \ldots, y_n} \perp$$

3. *(No proper ‘subselves’)*

   For any term $t$ that contains the variable $x$ but is not just $x$,

   $$t = x \vdash_{\text{FV}(t)} \perp$$
The classifying role of $\text{Sh}(\mathcal{L}, J_L)$

**Corollary**

If $L(X)$ is not constant then the topos $\text{Sh}(\mathcal{L}, \mathcal{K})$ classifies the theory presented by the sequents above together with:

4. (Basic operations are jointly surjective)

$$\top \vdash_x \bigvee_{n \in \mathbb{N}} \bigvee_{f \in \ell n} (\exists y_1, \ldots, y_n)(x = f(y_1, \ldots, y_n))$$

(this is a finite disjunction because $\ell n = \emptyset$ if $n > \text{deg } L(X)$).

Both the theorem and the corollary are essentially in Blass’ paper.
Theorem (Schanuel'91)

\[ \mathbb{N}[X, Y]/(X = 2X + 1, Y = X + 1 + Y, Y^2 = 2Y^2 + Y) \] is the Burnside rig of a preextensive category.
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\[ \mathbb{N}[X, Y]/(X = 2X + 1, \ Y = X + 1 + Y, \ Y^2 = 2Y^2 + Y) \text{ is the Burnside rig of a preextensive category. (Indeed, of the category of unbounded polyhedra.)} \]

Proof.

\[ X = (0, 1) \text{ and } Y = (0, \infty) \].
Theorem (Schanuel'91)

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Proof.

\[ X = (0, 1) \text{ and } Y = (0, \infty). \]

Free theories can be built for ‘multi-sorted signatures’.

Ample families make no reference to ‘sorts’.
Theorem

There exists a distributive category $E$ such that

$\mathcal{B}E = \mathbb{N}[X]/(X^2 = X + 1)$. 
1991 - Schanuel. *Negative sets have Euler characteristic and dimension.* LNM 1488.


1995 - Blass. *Seven trees in one.* JPAA.

1998 - Gates. *On the generic solution to* $P(X) = X$ *in distributive categories.* JPAA.


2000 - Schanuel. *Objective number theory and the retract chain condition.* (Montreal, 1997) JPAA.

2005 - Fiore & Leinster. *Objects of categories as complex numbers.* AIM.

2017 - Menni. *Every rig with a one-variable fixed point presentation is the Burnside rig of a preextensive category.* ACS.