On a problem in Objective Number Theory

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Objective Number Theory

"Diophantus probably considered natural numbers not in the abstract way which we habitually now do, but as born from actual objects. While the method of formally adjoining negatives and ascending to powerful cohomological calculations etc. leads to many results, we should not forget the objects themselves. Just as realizing cohomology classes by vector bundles via K-theory permitted powerful interplay between those calculations and directed manipulations of the objects by actual maps, so a similar possibility is opened by the Burnside rig of a distributive category, wherein polynomial equations satisfied by objects are revealed as specific structure on the objects themselves. [...]"

F. W. Lawvere. *Some thoughts on the future of category theory.* LNM 1488, 1991.

Distributive and extensive categories

A category with finite products and coproducts is said to be distributive if the canonical maps

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Proposition

Prextensive categories are distributive.

Rigs

Definition ("ring without negatives" (Schanuel'91))

A rig is a pair commutative monoids $(A, \cdot, 1)$ and $(A, +, 0)$ with the same underlying object and such that:

$$0 = a \cdot 0 \qquad (a \cdot b) + (a \cdot c) = a \cdot (b + c)$$

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1. Rings and Distributive lattices.
2. $\mathbb{N}$, $\mathbb{Q}_+$, $\mathbb{R}_+$.
3. The Burnside rig of isomorphism classes of objects in a prextensive category. (The additive structure never is a non-trivial group under.)

The Burnside rig of a distributive $\mathbf{E}$ will be denoted by $\mathfrak{B}\mathbf{E}$.

Negative sets have Euler characteristic and dimension

"**1. Where are the negative sets?**
Though ill-posed, the question is suggestive; [...] we seek an enlargement $\mathbb{E}$ [of the category of finite sets], the isomorphism classes of which should give rise to the integers, rather than just natural numbers."

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Theorem (Schanuel'91)

The induced $\mathbb{N}[X]/(X = 2X + 1) \to \mathfrak{B}(\mathbb{P}_0)$ *is an isomorphism.*

Proof)

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Theorem (Schanuel'91)

The induced $\mathbb{N}[X]/(X=2X+1) \to \mathfrak{B}(\mathbb{P}_0)$ *is an isomorphism.*

Proof) "Surjectivity is easy, because every bounded polyhedron is a disjoint union of open simpleces [...]. The heart of the matter is the injectivity of our map"

From *Some thoughts on the future of category theory*

"While the study of linear equations on distributive categories is packed with surprising subtleties, higher-degree equations are also approachable [...]

I was surprised to note that an isomorphism $x = 1 + x^2$ (leading to complex numbers as Euler characteristics if they don't collapse) always induces an isomorphism $x^7 = x$."

Blass and Gates

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Theorem (Gates'98)

For $L(X) \in \mathbb{N}[X]$ *having at least one constant term and at least one nonconstant term, there exists a prextensive category* $\mathbb{E}$ *such that* $\mathfrak{B}\mathbb{E} \cong \mathbb{N}[X]/(X = L(X))$.

Proof.

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Proof.

The hypotheses imply the existence of a faithful $\mathbb{E} \to \mathbf{Set}$. □

The original content of the talk

Theorem

For any $L(X) \in \mathbb{N}[X]$ *there exists a prextensive category* $\mathbb{E}$ *such that* $\mathfrak{B}\mathbb{E} \cong \mathbb{N}[X]/(X = L(X))$.

The case of constant polynomials is trivial.

The easy part

A prxtensive **E** and a surjective $\mathbb{N}[X]/(X = L(X)) \to \mathfrak{B}\mathbf{E}$.
(Using the strategy in Blass' paper.)

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Define $\mathsf{Fam}(\mathcal{T}, J) \to \mathsf{Sh}(\mathcal{T}, J)$ as the image of the top-right composite below

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so that $\mathrm{Fam}(\mathcal{T}, J)$ is prextensive and $\mathrm{Fam}(\mathcal{T}, J) \to \mathrm{Sh}(\mathcal{T}, J)$ is a full inclusion preserving finite products and finite coproducts.

(Free) Algebraic theories

$$\mathbf{Theories} \underset{\text{forget}}{\overset{\text{free}}{\leftrightarrows}} \mathbf{Set}^{\mathbb{N}} \quad \text{(free} \dashv \text{forget)}$$

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In general:

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In general:

$$L(X) \in \mathbb{N}[X] \longmapsto \ell \in \textbf{Set}^{\mathbb{N}} \overset{\text{free}}{\longmapsto} \mathcal{L} \in \textbf{Theories}$$

A coverage on a free algebraic theory

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Let J_L be the least Grothendieck topology on $\mathcal{L}$ such that T^1 is covered by the sieve generated by the family

$$(f : T^A \to T^1 \mid A \in \mathbf{fSet}, f \in \ell(\sharp A))$$

Example (For $L(X) = 1 + 2X^3$)

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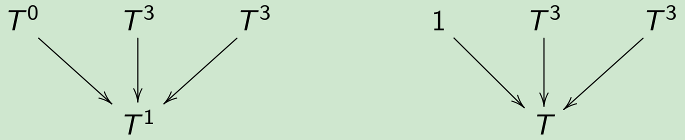

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Example (With $L(X) = 1 + 2X^3$)

$$1 \to T, \quad T^3 \to T, \quad T^3 \to T \qquad \text{in } \mathcal{L}$$

$$1 + T^3 + T^3 = L(T) \to T \qquad \text{in } \mathrm{Sh}(\mathcal{L}, J_L)$$

Basic operations are mono and disjoint

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Lemma

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Lemma

The canonical map $L(T) \to T$ in $\mathrm{Sh}(\mathcal{L}, J_L)$ is an iso.

Proof.

It is epic because it is induced by a covering family.

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It is epic because it is induced by a covering family.
It is monic because maps in $(f : T^A \to T \mid A \in \mathbf{fSet}, f \in \ell(\sharp A))$ are monic and 'disjoint' (in $\mathcal{L}$). □

So we have $\mathbb{N}[X]/(X = L(X)) \to \mathfrak{B}(\mathrm{Sh}(\mathcal{L}, J_L))$.

The canonical surjection

So we have $\mathbb{N}[X]/(X = L(X)) \to \mathfrak{B}(\mathsf{Sh}(\mathcal{L}, J_L))$. Moreover:

$$
\begin{array}{ccccc}
\mathbb{N}[X] & \twoheadrightarrow & \mathbb{N}[X]/(X = L(X)) & & \\
\cong \downarrow & & \vdots\downarrow & \searrow & \\
\mathfrak{B}(\mathsf{Fam}\mathcal{L}) & \twoheadrightarrow & \mathfrak{B}(\mathsf{Fam}(\mathcal{L}, J_L)) & \longrightarrow & \mathfrak{B}(\mathsf{Sh}(\mathcal{L}, J_L))
\end{array}
$$

so

$$\mathbb{N}[X]/(X = L(X)) \to \mathfrak{B}(\mathsf{Fam}(\mathcal{L}, J_L))$$

is surjective.

To prove that it is an iso, it is enough to build a retraction.

The not-so-easy part (sketch)

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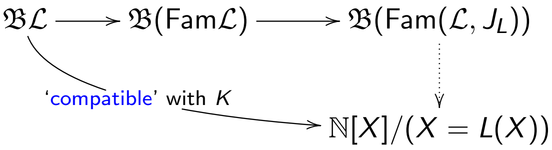

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\mathfrak{B}\mathcal{L} & \longrightarrow & \mathfrak{B}(\mathrm{Fam}\mathcal{L}) & \twoheadrightarrow & \mathfrak{B}(\mathrm{Fam}(\mathcal{L}, J_L)) \\
\cong \downarrow & & & & \vdots \\
(\mathbb{N}, +) & & \xrightarrow[\text{'compatible with } K\text{'}]{} & & \mathbb{N}[X]/(X = L(X)) \\
& & & & \\
n & & \longmapsto & & [X^n]
\end{array}$$

$$n \longmapsto [T^n]$$

END OF SKETCH.

Preliminaries to the reduction to bases. Terminology.

A family $(f_i : C_i \to C \mid i \in I)$ in a category $\mathcal{C}$ is called monic if the existence of a commutative diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{\;h\;} & C_j \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f_j} \\ C_i & \xrightarrow[\;f_i\;]{} & C \end{array}$$

implies $i = j$ and $g = h$.

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Lemma

A finite family $(f_i : C_i \to C \mid i \in I)$ is monic if and only if $[f_i \mid i \in I] : \sum_{i \in I} C_i \to C$ is monic in $\mathrm{Fam}\mathcal{C}$.

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A family is proper if it is finite and monic.

Compatibility for finite families

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Definition

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Following Blass, we could call $\sum_{i \in I} \gamma[C_i]$ the weight of the family $(f_i : C_i \to C \mid i \in I)$.

Example) $\rightsquigarrow$

Example, with $L(X) = 1 + X^2$

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is compatible with:

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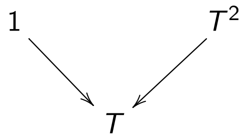

because

$$\gamma[1] + \gamma[T^2] = 1 + X^2 = X = \gamma[T]$$

Definition

A multiplicative $\gamma : \mathfrak{B}(\mathcal{L}) \to R$ is compatible with a basis K on $\mathcal{L}$ if γ is compatible with every K-cover.

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Proposition

If K is a 'good' basis on $\mathcal{C}$ and γ is compatible with it then there exists a unique $\gamma' : \mathfrak{B}(\mathrm{Fam}(\mathcal{C}, K)) \to R$ of rigs such that

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Let $\mathcal{C}$ be a small category with finite products.

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For $L(X) = 1 + X^2$.

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Lemma

The following four Grothendieck topologies on $\mathcal{L}$ coincide:

1. The smallest topology for which T is covered by the cospan $1 \to T \leftarrow T^2$.
2. The smallest topology in which, for each n, each T^k is covered by the set $S_n(T^k)$.
3. The topology where the covering sieves of any T^k are those sieves that include $S_n(T^k)$ for some n.
4. The topology where the covering sieves of any T^k are those sieves that include a finite family of maps $T^{k_i} \to T^k$ such that every map $1 \to T^k$ factors through a map from the finite family.

(It is part of the assertion of the lemma that the collections of sieves described in (3) and (4) are topologies.)

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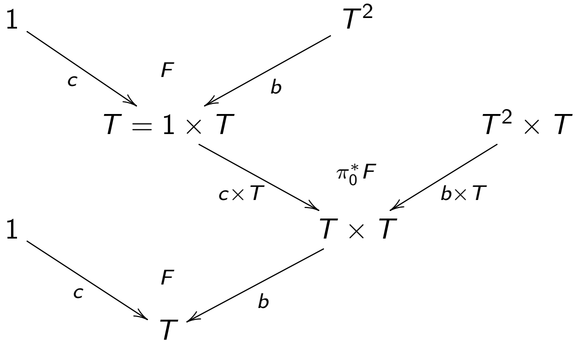

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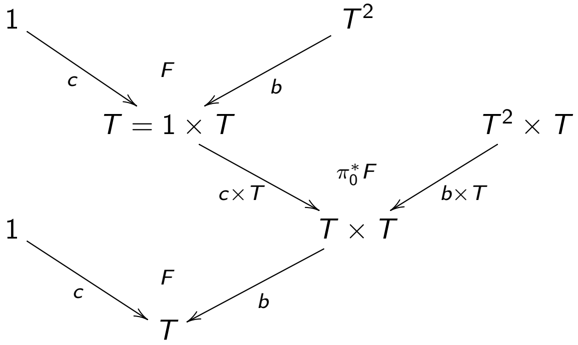

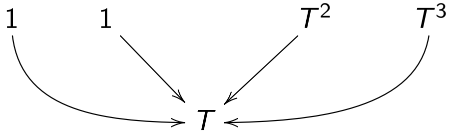

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Normed categories [Lawevere's 'Metric spaces' paper]

Let $(\mathcal{V}, \otimes, \mathbb{k})$ be a monoidal category.

Definition

A $\mathcal{V}$-normed category is a category $\mathcal{C}$ together with the assignment of an object $\mathfrak{d}_{X,Y} f = \mathfrak{d} f$ of $\mathcal{V}$ to every map $f : X \to Y$ in $\mathcal{C}$,

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The monoidal category $\mathbb{N}_\infty$

Let $(\mathbb{N}, +, 0)$ be the usual commutative monoid of natural numbers under addition and consider its extension $\mathbb{N}_\infty = (\mathbb{N} + \{\infty\}, +, 0)$ with an element ∞ such that, for every $n \in \mathbb{N} + \{\infty\}$, $\infty + n = \infty = \infty + n$.

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The monoid structure induces a total order $(\mathbb{N}_\infty, \leq)$ with ∞ as terminal object. Moreover, addition extends to a symmetric monoidal structure on the category $(\mathbb{N}_\infty, \leq)$. The resulting monoidal category $((\mathbb{N}_\infty, \leq), +, 0)$ will be denoted simply by $(\mathbb{N}_\infty, \leq)$.

Categories normed in $(\mathbb{N}_\infty, \leq)$

In concrete terms, an $(\mathbb{N}_\infty, \leq)$-normed category is a category $\mathcal{C}$ equipped with a collection $(\mathfrak{d}_{X,Y} : \mathcal{C}(X, Y) \to \mathbb{N}_\infty \mid X, Y \in \mathcal{C})$ of functions such that:

$$(\mathfrak{d}g) + (\mathfrak{d}f) \leq \mathfrak{d}(gf)$$

holds for every $f : X \to Y$ and $g : Y \to Z$.
(It automatically holds that

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(If the $\mathcal{C}$ has a terminal object then we may entertain the condition that $\mathfrak{d}p = \infty$ for every point $p : 1 \to C$.)

Examples of normed categories

Let $L(X) \in \mathbb{N}[X]$ and let $\mathcal{L}$ be the resulting free theory.

Proposition

The assignment $\eth$ that sends

1. each basic constant $1 \to T$ in $\mathcal{L}$ to $\infty \in \mathbb{N}_\infty$ and
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Intuition: if you think of a map $f : T^A \to T$ in $\mathcal{L}$ as a tree with leaves in A then $\mathfrak{d}f$ is the length of a shortest path to a leaf.

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Ample families

Let $\mathcal{C}$ be an $(\mathbb{N}_\infty, \leq)$-normed category with 'norm' $\mathfrak{d}$.

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Let $\mathfrak{K}Y$ be the set of ample and proper families with codomain Y.

Is $\mathfrak{K}$ the basis for a Grothendieck topology?

A generalization of Blass' Lemma

Let $L(X) \in \mathbb{N}[X]$.
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Proper and strictly n-ample families are unique up to iso in the following sense.

Lemma

If $F = (f_i : X_i \to Y \mid i \in I)$ and $H = (h_j : A_j \to Y \mid j \in J)$ are both strictly n-ample and proper families then there exists a unique bijection $\phi : I \to J$ such that for every $i \in I$, $h_{\phi i}$ is iso to f_i over Y.

Blass' Lemma again

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The main result

Theorem

For any $L(X) \in \mathbb{N}[X]$, *the canonical* $\mathbb{N}[X]/(X = L(X)) \to \mathfrak{B}(\mathrm{Fam}(\mathcal{L}, J_L))$ *is an isomorphism.*

Proof.

For non-constant $L(X) \in \mathbb{N}[X]$, $\mathrm{Fam}(\mathcal{L}, J_L) \cong \mathrm{Fam}(\mathcal{L}, \mathfrak{K})$ by the 'ample' Blass Lemma.

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$$\mathfrak{B}\mathcal{L} \xrightarrow{\cong} \mathbb{N} \longrightarrow \mathbb{N}[X]/(X = L(X))$$

is compatible with $\mathfrak{K}$, so it extends 'along $\mathcal{L} \to \mathrm{Fam}(\mathcal{L}, \mathfrak{K})$' to a

$$\mathfrak{B}(\mathrm{Fam}(\mathcal{L}, \mathfrak{K})) \to \mathbb{N}[X]/(X = L(X))$$

that is a retraction to $\mathbb{N}[X]/(X = L(X)) \to \mathfrak{B}(\mathrm{Fam}(\mathcal{L}, J_L))$. □

Optional

Developments and compatibility

Pullbacks of the basic family

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Let B be a family of maps in $\mathcal{T}$ with codomain T.

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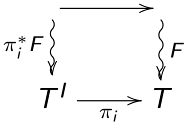
For any finite I and $i \in I$, let $\pi_i^* F$ the pullback family



Families of the form $\pi_i^* F$ will be called primitive.

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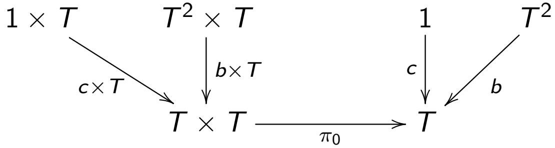
Let $\mathcal{L}$ be the free theory on $c : 1 \to T$ and $b : T^2 \to T$.
Let $F = \{c, b\}$ the family on the right below.
So that $\pi_0^* F$ is the family on the left below.



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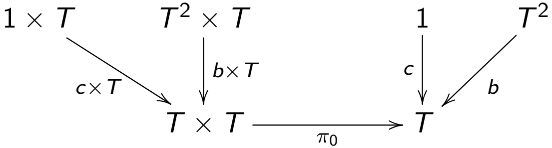
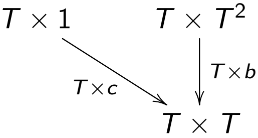


Similarly, $\pi_1^* F$ is



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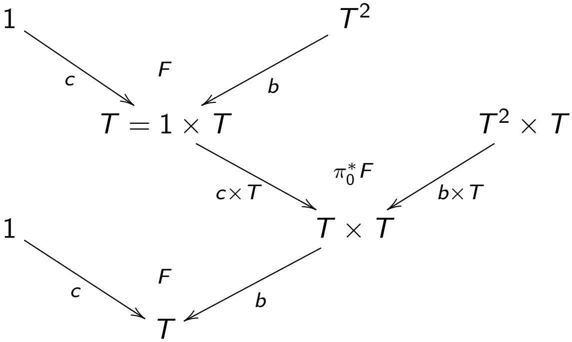

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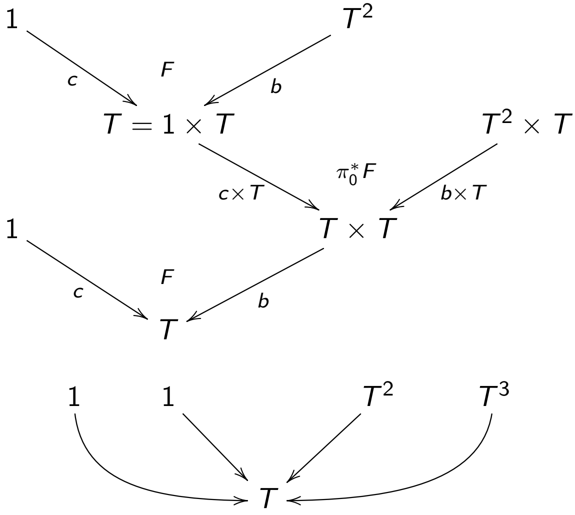
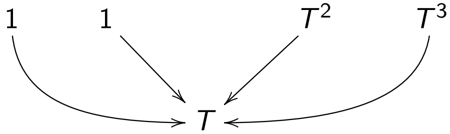


$c \qquad b(c, c) \qquad b(c, b(x, y)) \qquad b(b(x, y), z)$

Slogan/Theorem

Developments make sense in any algebraic theory equipped with a distinguished ‘cover’.

Compatibility for bases (again)



Let $\gamma : \mathfrak{B}\mathcal{L} \to R = \mathbb{N}[X]/(X = 1 + X^2)$ be the morphism of multiplicative monoids senting $[T]$ to X.

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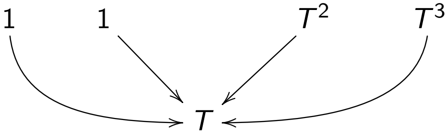

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Recall!! γ is compatible with $(f_i : C_i \to C \mid i \in I)$,
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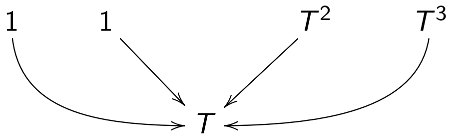

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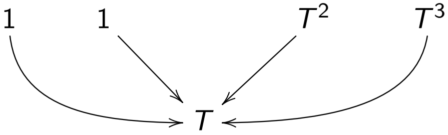

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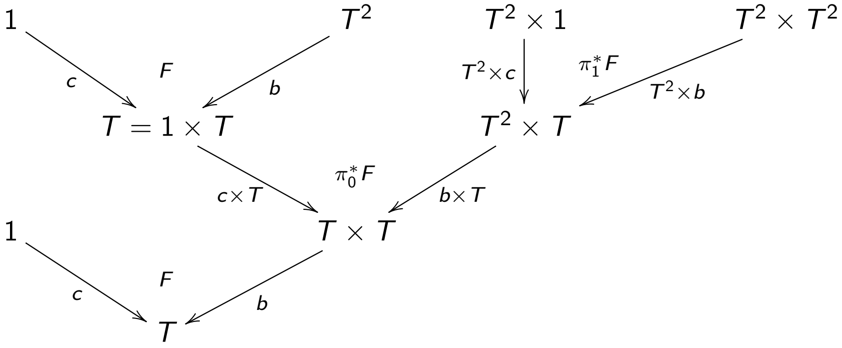
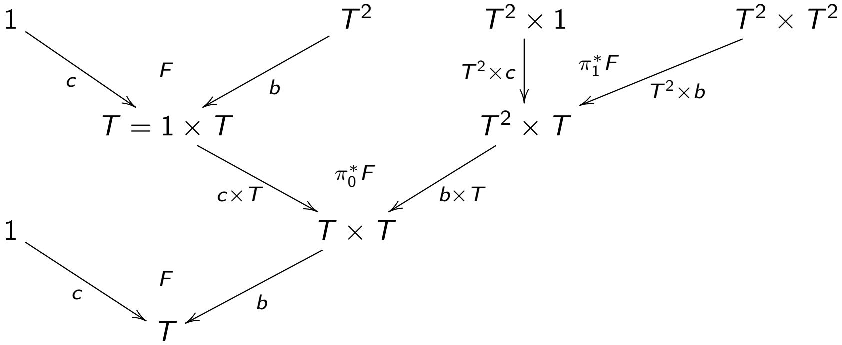
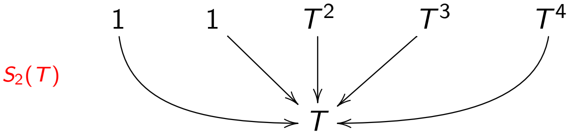
Slogan/Theorem) If γ is compatible with F then it is compatible with any development of F.

Developments to level 2



Developments to level 2





$c \qquad b(c, c) \qquad b(c, b(x, y)) \qquad b(b(x, y), c) \qquad b(b(x, y), b(x', y'))$

Further comments

The classifying role of $\mathsf{Sh}(\mathcal{L}, J_L)$

Theorem

If $\mathcal{T}$ is the free algebraic theory generated by $\ell \in \mathbf{Set}^{\mathbb{N}}$. Then $\widehat{\mathcal{T}}$ classifies the theory presented by the same operations and

1. *(Basic ops are injective) For every $n \in \mathbb{N}$ and $f \in \ell n$:*

$$f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n) \vdash_{x_1,\ldots,x_n,y_1,\ldots,y_n} \bigwedge_{i=1}^{n} x_i = y_i$$

2. *(Basic ops are disjoint) For all $f \in \ell m$, $f' \in \ell n$ and $f \neq f'$:*

$$f(x_1, \ldots, x_m) = f'(y_1, \ldots, y_n) \vdash_{x_1,\ldots,x_m,y_1,\ldots,y_n} \bot$$

3. *(No proper 'subselves')*
   For any term t that contains the variable x but is not just x,

$$t = x \vdash_{FV(t)} \bot$$

The classifying role of $\mathrm{Sh}(\mathcal{L}, J_L)$

Corollary

If $L(X)$ is not constant then the topos $\mathrm{Sh}(\mathcal{L}, \mathfrak{K})$ classifies the theory presented by the sequents above together with:

4. *(Basic operations are jointly surjective)*

$$\top \vdash_x \bigvee_{n \in \mathbb{N}} \bigvee_{f \in \ell n} (\exists y_1, \ldots, y_n)(x = f(y_1, \ldots, y_n))$$

(this is a finite disjunction because $\ell n = \emptyset$ if $n > \deg L(X)$).

Both the theorem and the corollary are essentially in Blass' paper.

Systems of fix-point equations

Theorem (Schanuel'91)

$\mathbb{N}[X, Y]/(X = 2X + 1, Y = X + 1 + Y, Y^2 = 2Y^2 + Y)$ *is the Burnside rig of a prextensive category.*

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Proof.

$X = (0,1)$ and $Y = (0,\infty)$. □

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Free theories can be built for 'multi-sorted signatures'.

Ample families make no reference to 'sorts'.

mystification and categorification

From: Stephen Schanuel
Subject: mystification and categorification
Newsgroups: gmane.science.mathematics.categories
Date: Thursday 4th March 2004 05:44:46 UTC

Theorem

There exists a distributive category $\mathbf{E}$ *such that*
$\mathfrak{B}\mathbf{E} = \mathbb{N}[X]/(X^2 = X + 1)$.

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