String Diagrams for (Virtual) Proarrow Equipments

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Theorem (Joyal and Street)

The graphical calculus for monoidal categories is sound.

For any deformation \( h : \Gamma \times [0, 1] \rightarrow [a, b] \times [c, d] \) of diagrams, the value of \( h(\cdot, 0) \) equals that of \( h(\cdot, 1) \).
Theorem (M.)

The graphical calculi for double categories and equipments are sound.

For any deformation $h: \Gamma \times [0, 1] \to [a, b] \times [c, d]$ of diagrams, the value of $h(-, 0)$ equals that of $h(-, 1)$. 
Theorem (M.)

There is a canonical (Yoneda-style) embedding $| \cdot | : \mathcal{E} \rightarrow \mathcal{E}\text{-Cat}$ of a virtual equipment into the virtual equipment of categories enriched in it, which is full on arrows and coreflective on proarrows.
What is a Double Category

A **double category** is a category internal to the category of categories.
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- Objects $A, B, \ldots,$
- Arrows $f : A \to B, \ldots$
- Proarrows $J : A \to B, \ldots$
- 2-cells $\ldots$
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- **Arrows** $f : A \to B, \ldots, \ldots$

- **Proarrows** $J : A \rightarrowtail B, \ldots, \ldots$

- **2-cells** $\alpha$, $C \rightarrow D$, $\ldots, \ldots$
What is a Double Category
Companions and Conjoint

An arrow $\ydiagram$ has a *companion* if there is a proarrow $\xymatrix@C=1.5em{&\bullet\ar@{..}@/_/[r]\ar@{..}@/^/[r]}$ together with two 2-cells $\alpha$ and $\beta$ such that $\alpha = \beta = \gamma$.

Similarly, $\xymatrix@C=1.5em{&\bullet\ar@{..}@/_/[r]\ar@{..}@/^/[r]}$ is said to have a *conjoint* if there is a proarrow $\ydiagram$ together with two 2-cells $\alpha$ and $\beta$ such that $\alpha = \beta = \gamma$. 
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An arrow has a *companion* if there is a proarrow together with two 2-cells and
Companions and Conjoints

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\[ \begin{align*}
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}
\end{align*}
\]

and

\[ \begin{align*}
\begin{array}{c}
\text{Diagram 3} \\
\text{Diagram 4}
\end{array}
\end{align*}
\]
Companions and Conjoints

An arrow \(ightarrow\) has a *companion* if there is a proarrow \(\rightarrow\) together with two 2-cells \(\Rightarrow\) and \(\Rightarrow\) such that

\[
\begin{align*}
\begin{array}{cc}
\Rightarrow & = \\
\end{array}
\quad \text{and} \quad \\
\begin{array}{cc}
\Rightarrow & = \\
\end{array}
\end{align*}
\]

Similarly, \(\rightarrow\) is said to have a *conjoint* if there is a proarrow \(\rightarrow\)
Companions and Conjoints

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\text{arrow} \\
\end{array} \) has a *companion* if there is a proarrow \( \begin{array}{c}
\text{proarrow} \\
\end{array} \) together with two 2-cells \( \begin{array}{c}
\text{2-cell} \\
\end{array} \) and \( \begin{array}{c}
\text{2-cell} \\
\end{array} \) such that

\[
\begin{array}{c}
\text{2-cell} \\
\end{array} \quad = \quad \begin{array}{c}
\text{2-cell} \\
\end{array} \\
\text{and} \\
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Similarly, \( \begin{array}{c}
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\text{Diagram 4}
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\end{align*}
\]

Similarly, is said to have a *conjoint* if there is a proarrow together with two 2-cells and such that

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 5} \\
\text{Diagram 6}
\end{array}
\end{align*}
\]

and

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 7} \\
\text{Diagram 8}
\end{array}
\end{align*}
\]
Definition

A **proarrow equipment** is a double category where every arrow has a conjoint and a companion.
Proarrow Equipments

Definition
A proarrow equipment is a double category where every arrow has a conjoint and a companion.

Examples

- Sets, Functions, Relations.
- Rings, Homomorphisms, Bimodules.
- Categories, Functors, Profunctors.
- Enriched Categories, Enriched Functors, Enriched Profunctors, etc.
Lemma (Spider Lemma)

In an equipment, we can bend arrows. More formally, there is a bijective correspondence between diagrams of form of the left, and diagrams of the form of the right:
Hom-Set to Zig-Zag Adjunctions

Given arrows \[ \quad \text{and} \quad \]

\[ \quad \]
Hom-Set to Zig-Zag Adjunctions

Given arrows and , with an isomorphism with

inverse:
Given arrows \( \begin{array}{c} \downarrow \end{array} \) and \( \begin{array}{c} \downarrow \end{array} \), with an isomorphism \( \begin{array}{c} \leftarrow \rightarrow \end{array} \) with inverse \( \begin{array}{c} \rightarrow \leftarrow \end{array} \) and \( \begin{array}{c} \rightarrow \leftarrow \end{array} \) and \( \begin{array}{c} \rightarrow \leftarrow \end{array} \) and \( \begin{array}{c} \rightarrow \leftarrow \end{array} \).
Hom-Set to Zig-Zag Adjunctions

Given arrows \( \text{and} \), with an isomorphism with

inverse

and

Bend to and
Hom-Set to Zig-Zag Adjunctions

Given arrows \( \begin{tikzpicture} \draw[->,thick] (0,0) .. controls (0.5,0) and (-0.5,0) .. (0,0); \end{tikzpicture} \) and \( \begin{tikzpicture} \draw[->,thick] (0,0) .. controls (0.5,0) and (-0.5,0) .. (0,0); \end{tikzpicture} \), with an isomorphism \( \begin{tikzpicture} \draw[->,thick] (0,0) .. controls (0.5,0) and (-0.5,0) .. (0,0); \end{tikzpicture} \) with inverse \( \begin{tikzpicture} \draw[->,thick] (0,0) .. controls (0.5,0) and (-0.5,0) .. (0,0); \end{tikzpicture} \) and \( \begin{tikzpicture} \draw[->,thick] (0,0) .. controls (0.5,0) and (-0.5,0) .. (0,0); \end{tikzpicture} \).

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\[ \begin{array}{c}
\begin{tikzpicture} \draw[->,thick] (0,0) .. controls (0.5,0) and (-0.5,0) .. (0,0); \end{tikzpicture} \quad = \quad \begin{tikzpicture} \draw[->,thick] (0,0) .. controls (0.5,0) and (-0.5,0) .. (0,0); \end{tikzpicture} \\
\begin{tikzpicture} \draw[->,thick] (0,0) .. controls (0.5,0) and (-0.5,0) .. (0,0); \end{tikzpicture} \quad = \quad \begin{tikzpicture} \draw[->,thick] (0,0) .. controls (0.5,0) and (-0.5,0) .. (0,0); \end{tikzpicture}
\end{array} \]
Zig-Zag to Hom-Set Adjunctions

Given \( s \) and \( t \), satisfying \( s = s' \) and \( t = t' \), then \( s'' = s''' = t'' = t''' \), \( s'' = s''' = t'' = t''' \),
Zig-Zag to Hom-Set Adjunctions

Given \( \natural X \) and \( \natural Y \), satisfying

\[
\begin{align*}
\natural X &= \cdots \\
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\]

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Lawvere (’73):

- Not only are the most fundamental structures of mathematics organized in categories,
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Not only are the most fundamental structures of mathematics organized in categories,
They are in many cases (enriched) categories themselves.

With the graphical calculus, we can show that so long as our objects form a virtual equipment, then they are enriched categories of a sort.

Theorem (M.)

There is a Yoneda-style embedding \(|\cdot|: \mathcal{E} \to \mathcal{E}\text{-Cat}\) of a virtual equipment into the virtual equipment of categories enriched in it.
Enrichment and Virtual Equipments

- Composing proarrows requires taking a colimit in the base category.
- But what if the base category is not suitably cocomplete?
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But what if the base category is not suitably cocomplete?

Then we use “virtual equipments” instead.
Definition

A cell is called *cartesian* if for any unique so that

We call the *restriction* of along and .
Restrictions

Definition
A **Virtual Equipment** is a virtual double category with all restrictions (and a unit condition).

Lemma (Cruttwell and Shulman)
*In a virtual equipment, every restriction is of the form*
Enriching in a Virtual Equipment

The main difference between enriching in a virtual equipment and enriching in a monoidal category is extent:
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\[ C \text{ a } \mathcal{V}\text{-category means: } \forall A, B \in C_0, \quad C(A, B) \in \mathcal{V}, \]
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Examples of Enrichment in a Virtual Equipment

With a single object:

- In Sets and Spans: Categories.
- In Rings and Bimodules: Algebras.
- In Enriched Cats and Profunctors: Arrows.
- Multicategories, Many-sorted Lawvere theories, Virtual double categories, etc.
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With many objects:
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Conjecture (M.)

There is a full and faithful functor

\[ \text{Kleisli}(\text{Jet}) \hookrightarrow \text{Span-Cat}. \]

sending a smooth manifold to its category of smooth paths.
Enriching in a Virtual Equipment

A category $\mathcal{C}$ enriched in a virtual equipment $\mathcal{E}$ consists of the following data:

- A class of objects $\mathcal{C}_0$, with each object $A \in \mathcal{C}_0$ associated with an object $\mathcal{C}(A) = \square$ in $\mathcal{E}$ called its extent.
- For each pair of objects $\square$ and $\square$ in $\mathcal{C}_0$, a proarrow $\mathcal{C}(\square, \square) = \square$ in $\mathcal{E}$.
- For each object $\square$ in $\mathcal{C}_0$, a 2-cell $\text{id}_\square = \square$ called the identity.
- For each triple of objects $\square$, $\square$, $\square$, a 2-cell called composition.
Defining the “Yoneda” Embedding

For an object $\text{of } \mathcal{E}$, we define its representative to be

\[
\begin{align*}
\text{Objects are vertical arrows, with each object’s extent being its domain.} \\
\text{Between objects and , a hom-object.} \\
\text{For object , an identity arrow.} \\
\text{For each composable triple, a composition arrow.}
\end{align*}
\]
Defining the “Yoneda” Embedding
Properties of “Yoneda” Embedding

Proposition (M.)

The “Yoneda” embedding \(| \cdot | : \mathcal{E} \rightarrow \mathcal{E}\text{-Cat}\)

- is full on 2-cells (and therefore faithful on arrows and proarrows);
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- preserves composition;
- reflects Morita equivalence.
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- reflects Morita equivalence.

Conjecture (M.)

For a “fibrantly enriched” $\mathcal{E}$-category $\mathcal{C}$, denote by $\mathcal{C}[A]$ the full subcategory of $\mathcal{C}$ whose objects have extent $A$. Then

$$\mathcal{E}\text{-Cat}(|A|, \mathcal{C}) \simeq \mathcal{C}[A].$$
Soundness of Graphical Calculi

- Similar to the proof of Joyal and Street for monoidal categories.
- But using the tile-order machinery of Dawson and Paré to handle the two sorts of composition.
Take a Diagram,
Label it,
Tile it,
Turn it into the usual notation,
Compose it.
Tilings are stable under small deformations.
In Conclusion

Equipments are fundamental and useful objects
1. for combining “scalar” arrows and “linear” proarrows, and
2. as a setting for formal (enriched, internal, higher) category theory.

I hope that the string diagrams can make working with them easier!
Acknowledgements: Many thanks to Emily Riehl and Mike Shulman for reading drafts and giving very helpful comments.


2. Joyal and Street, *Geometry of tensor calculus, I*.

3. Dawson and Paré, *General associativity and general composition for double categories*


5. Cruttwell and Shulman, *A unified framework for generalized multicategories*

6. Leinster, *Generalized enrichment of categories*