Topological Groupoids and Exponentiability

Susan Niefield
(joint with Dorette Pronk)

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Overview

Goal: Study exponentiability in categories of topological groupoid.

Starting Point: Consider exponentiability in $\mathbf{Gpd}(\mathcal{C})$, where $\mathcal{C}$ is:

- finitely complete
- cartesian closed
- locally cartesian closed

Application: Adapt to various categories of orbigroupoids.

Exponentiability in $\mathcal{C}$

Suppose $\mathcal{C}$ is finitely complete. An object Y is called *exponentiable* if $- \times Y : \mathcal{C} \longrightarrow \mathcal{C}$ has a right adjoint, and $\mathcal{C}$ is called *cartesian closed* if every object is exponentiable.

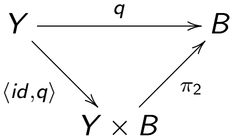
A morphism $Y \longrightarrow B$ is *exponentiable* in $\mathcal{C}$ if it is exponentiable in the slice category $\mathcal{C}/B$, and $\mathcal{C}$ is called *locally cartesian closed* if every morphism is exponentiable.

Note that if $q\colon Y \longrightarrow B$ is exponentiable and $r\colon Z \longrightarrow B$, we follow the abuse of notation and write the exponential as $r^q\colon Z^Y \longrightarrow B$.

Properties of Exponentiability

Composition of exponentiables is exponentiable, and pullback along any morphism preserves exponentiability.

If the diagonal $B \xrightarrow{\Delta} B \times B$ and Y are exponentiable, then every morphism $Y \xrightarrow{q} B$ is exponentiable, since



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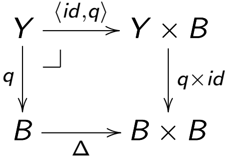

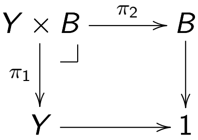

Exponentiable Spaces

A space Y is exponentiable in **Top** iff $\mathcal{O}(Y)$ is a continuous lattice (Day/Kelly, 1970)

A sober space is exponentiable in **Top** iff it is locally compact (Hoffmann/Lawson, 1978)

A subspace inclusion is exponentiable in **Top** iff it is locally closed (N, 1978)

Cartesian Closed Coreflective Subcategories of **Top**

Suppose $\mathcal{M} \subseteq \mathbf{Top}$. Given a space X, let $\hat{X}$ denote the set X with the topology generated by the set of continuous maps

$$\{f\colon M \rightarrow X \mid M \in \mathcal{M}\}$$

Say a X is $\mathcal{M}$-*generated* if $X = \hat{X}$, and let $\mathbf{Top}_{\mathcal{M}}$ denote the full subcategory of **Top** consisting of $\mathcal{M}$-generated space.

Proposition (N, 1978) If $\mathcal{M}$ is a class of exponentiable spaces s.t. $M \times N \in \mathbf{Top}_{\mathcal{M}}$, for all $M, N \in \mathcal{M}$, then $\mathbf{Top}_{\mathcal{M}}$ is cartesian closed.

Note $X \hat{\times} Y = \widehat{X \times Y}$ is the product and $Z^Y = \widehat{\lim_{M \rightarrow Y} Z^M}$ is the exponential in $\mathbf{Top}_{\mathcal{M}}$.

Examples

$\mathcal{K}$ = compact T_2 spaces; $\mathbf{Top}_{\mathcal{K}}$ = compactly generated spaces

$\mathcal{E}$ = exponentiable spaces; $\mathbf{Top}_{\mathcal{E}}$ = exponentiably generated spaces

Note In both cases, one can show locally closed inclusions are exponentiable in $\mathbf{Top}_{\mathcal{M}}$. Thus, if $\Delta\colon B \longrightarrow B \hat{\times} B$ is locally closed, then the slice $\mathbf{Top}_{\mathcal{M}}/B$ is cartesian closed.

Groupoids in $\mathcal{C}$

An object G of $\mathbf{Gpd}(\mathcal{C})$ is a diagram in $\mathcal{C}$ of the form

$$G_2 \xrightarrow{\;c\;} G_1 \overset{i}{\circlearrowright} \begin{array}{c} \xrightarrow{\;s\;} \\ \xleftarrow{\;e\;} \\ \xrightarrow[\;t\;]{} \end{array} G_0$$

making G a category in $\mathcal{C}$ in which every morphism is invertible, where $G_2 = G_1 \times_{G_0} G_1$. Unless otherwise stated, $G_1 \longrightarrow G_0$ is t in the pullback when G_1 is on the left and s when it is on the right.

Morphisms $f\colon G \longrightarrow H$ are pairs $f_i\colon G_i \longrightarrow H_i$ $(i = 0, 1)$ compatible with the groupoid structure, i.e., *homomorphisms*.

2-Cells $f \Rightarrow g\colon G \longrightarrow H$ are morphisms $\varphi\colon G_0 \longrightarrow H_1$ such that

$$\begin{array}{ccc} G_1 & \xrightarrow{\langle \varphi s, g_1 \rangle} & H_2 \\ {\scriptstyle \langle f_1, \varphi t \rangle}\downarrow & & \downarrow{\scriptstyle c} \\ H_2 & \xrightarrow[c]{} & H_1 \end{array}$$

i.e., *natural transformations*.

Exponentiable Objects in $\mathbf{Gpd}(\mathcal{C})$

$\mathbf{Gpd}(\mathcal{C})$ is cartesian closed whenever $\mathcal{C}$ is, and H^G is defined by

$$(H^G)_0 \rightarrowtail H_0^{G_0} \times H_1^{G_1} \underset{g_0}{\overset{f_0}{\rightrightarrows}} X_0$$

$$(H^G)_1 \rightarrowtail (H^G)_0 \times H_1^{G_0} \times (H^G)_0 \underset{g_1}{\overset{f_1}{\rightrightarrows}} X_1$$

where (f_0, g_0) and (f_1, g_1) make H^G the "groupoid of homomorphisms" $G \longrightarrow H$.

In particular, $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{K}})$ and $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{E}})$ are cartesian closed.

Exponentiable Objects in $\mathbf{Gpd}(\mathcal{C})$, cont.

This construction of H^G uses only the fact that G_0, G_1, and G_2 are exponentiable and not that $\mathcal{C}$ is cartesian closed, and so:

Proposition If G_0, G_1, and G_2 are exponentiable in $\mathcal{C}$, then G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$.

Proposition If G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$, then G_0 is exponentiable in $\mathcal{C}$. The converse holds if s or t is exponentiable in $\mathcal{C}$.

Note If G is an étale groupoid, then all structure morphisms are exponentiable in **Top**. We conjecture that, in this case, H^G is étale when H is also étale and G_1/G_0 is compact.

Example

If $\mathcal{C}$ also has finite coproducts, we can consider $\mathbb{I}\colon 0 \xrightarrow{\cong} 1$, which is exponentiable in $\mathbf{Gpd}(\mathcal{C})$.

$(B^{\mathbb{I}})_0 = B_1$, the "object of morphisms $\beta\colon b \longrightarrow \bar{b}$"

$(B^{\mathbb{I}})_1 = B_2 \times_{B_1} B_2$ via c, the "object of squares"

$$\begin{array}{ccc} b_s & \xrightarrow{\alpha} & b_t \\ {\scriptstyle \beta_s}\downarrow & & \downarrow{\scriptstyle \beta_t} \\ \bar{b}_s & \xrightarrow[\bar{\alpha}]{} & \bar{b}_t \end{array}$$

$$\text{with} \quad s(\beta_s \overset{\alpha}{\underset{\bar{\alpha}}{\rightrightarrows}} \beta_t) = \beta_s, \;\; t(\beta_s \overset{\alpha}{\underset{\bar{\alpha}}{\rightrightarrows}} \beta_t) = \beta_t, \;\; \ldots$$

$(B^{\mathbb{I}})_2$ is the "object of horizontally composable squares"

Note $B^{\mathbb{I}}$ is an orbigroupoid when B is.

Exponentiable Morphisms in **Gpd**($\mathcal{C}$)

$G \xrightarrow{q} B$ is a *fibration* if $G_1 \xrightarrow{\langle s, q_1 \rangle} G_0 \times_{B_0} B_1$ has a right inverse in $\mathcal{C}$.

If $\mathcal{C}$ is locally cartesian closed, then every fibration $G \xrightarrow{q} B$ is exponentiable in **Gpd**($\mathcal{C}$) and $r^q \colon H^G \longrightarrow B$ is defined by

$$(H^G)_0 \rightarrowtail H_0^{G_0} \times_{B_0} (B_0 \times_{B_1} H_1^{G_1}) \underset{g_0}{\overset{f_0}{\rightrightarrows}} X_0$$

$$(H^G)_1 \rightarrowtail (H^G)_0 \times_{B_0} H_1^{G_1} \times_{B_0} (H^G)_0 \underset{g_1}{\overset{f_1}{\rightrightarrows}} X_1$$

making $(H^G)_0$ "the object of homomorphisms $G_b \xrightarrow{\sigma} H_b$," and $(H^G)_1$ "the object of 2-cells between distributers"

$$\begin{array}{ccc} G_b & \xrightarrow{\sigma} & H_b \\ {\scriptstyle G_\beta}\big\downarrow\!\bullet & \xrightarrow{\Sigma} & \bullet\!\big\downarrow{\scriptstyle H_\beta} \\ G_{\bar{b}} & \xrightarrow[\bar{\sigma}]{} & H_{\bar{b}} \end{array}$$

Note Composition is defined using the fibration assumption.

Exponentiable Morphisms in $\mathbf{Gpd}(\mathcal{C})$, cont.

The construction of H^G uses only that $G \xrightarrow{q} B$ is a fibration with the q_i exponentiable, and not that $\mathcal{C}$ is locally cartesian closed.

Proposition If $G \longrightarrow B$ is a fibration with $G_i \longrightarrow B_i$ $(i = 0, 1, 2)$ exponentiable in $\mathcal{C}$, then $G \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})$.

Corollary Suppose the diagonals $B_i \xrightarrow{\Delta_i} B_i \times B_i$ $(i = 0, 1)$ are exponentiable in a cartesian closed category $\mathcal{C}$. Then every fibration $G \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})$

Note This implies that fibrations $G \longrightarrow B$ are exponentiable in $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{M}})$, for $\mathcal{M} = \mathcal{K}$ or $\mathcal{E}$, if $B_i \xrightarrow{\Delta_i} B_i \hat{\times} B_i$ are locally closed.

Example Revisited

Define $B^{\mathbb{I}} \overset{s}{\underset{t}{\rightrightarrows}} B$ by $s_0 = s$, $t_0 = t$, $s_1 \colon B_2 \times_{B_1} B_2 \xrightarrow{\pi_2} B_2 \xrightarrow{\pi_1} B_1$, and $t_1 \colon B_2 \times_{B_1} B_2 \xrightarrow{\pi_1} B_2 \xrightarrow{\pi_2} B_1$, i.e., $s_1(\beta_s \overset{\alpha}{\underset{\bar{\alpha}}{\rightrightarrows}} \beta_t) = \alpha$, $t_1(\beta_s \overset{\alpha}{\underset{\bar{\alpha}}{\rightrightarrows}} \beta_t) = \bar{\alpha}$.

Proposition $B^{\mathbb{I}} \overset{s}{\underset{t}{\rightrightarrows}} B$ are fibrations in $\mathbf{Gpd}(\mathcal{C})$.

Proof. $\langle s, s_1 \rangle \colon (B^{\mathbb{I}})_1 \longrightarrow (B^{\mathbb{I}})_0 \times_{B_0} B_1$ is given by

$$\begin{array}{ccc} b_s & \xrightarrow{\alpha} & b_t \\ {\scriptstyle\beta_s}\downarrow & & \downarrow{\scriptstyle\beta_t} \\ \bar{b}_s & \xrightarrow[\bar{\alpha}]{} & \bar{b}_t \end{array} \;\mapsto\; \begin{array}{ccc} b_s & \xrightarrow{\alpha} & b_t \\ {\scriptstyle\beta_s}\downarrow & & \\ \bar{b}_s & & \end{array} \quad \text{and so} \quad \begin{array}{ccc} b_s & \xrightarrow{\alpha} & b_t \\ {\scriptstyle\beta_s}\downarrow & & \\ \bar{b}_s & & \end{array} \;\mapsto\; \begin{array}{ccc} b_s & \xrightarrow{\alpha} & b_t \\ {\scriptstyle\beta_s}\downarrow & & \downarrow{\scriptstyle id} \\ \bar{b}_s & \xrightarrow[\alpha\beta_s^{-1}]{} & b_t \end{array}$$

is a right inverse. The proof for t is similar.

Note Can also show $B^{\mathbb{I}} \times_B G \xrightarrow{s\pi_1} B$ is a fibration, for all $G \longrightarrow B$.

Example Revisited, cont.

Proposition $B^{\mathbb{I}} \times_B G \xrightarrow{s\pi_1} B$ is exponentiable in **Gpd**$(\mathcal{C})$, if the components are exponentiable in $\mathcal{C}$.

Corollary Suppose the diagonals $B_i \xrightarrow{\Delta_i} B_i \times B_i$ $(i = 0, 1)$ are exponentiable in a cartesian closed category $\mathcal{C}$. Then $B^{\mathbb{I}} \times_B G \xrightarrow{s\pi_1} B$ is exponentiable, for all $G \longrightarrow B$ in **Gpd**$(\mathcal{C})$.

Note So, $B^{\mathbb{I}} \times_B G \xrightarrow{s\pi_1} B$ is exponentiable in **Gpd**$(\mathbf{Top}_{\mathcal{M}})$, for $\mathcal{M} = \mathcal{K}$ or $\mathcal{E}$, if the diagonals $B_i \xrightarrow{\Delta_i} B_i \hat{\times} B_i$ are locally closed.

The Pseudo-Slice 2-Category $\mathbf{Gpd}(\mathcal{C})//B$

The functor $(G \longrightarrow B) \mapsto (B^{\mathbb{I}} \times_B G \xrightarrow{s\pi_1} B)$ is part of a 2-monad on the 2-category $\mathbf{Gpd}(\mathcal{C})/B$ induced by the internal groupoid

$$B^{\mathbb{I}} \times_B B^{\mathbb{I}} \xrightarrow{c} B^{\mathbb{I}} \overset{i}{\circlearrowleft} \quad \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{e} \\ \xrightarrow[t]{} \end{array} B$$

and the 2-Kleisli category is the pseudo-slice $\mathbf{Gpd}(\mathcal{C})//B$ whose objects are homomorphism $q\colon G \longrightarrow B$, morphisms are triangles

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ & \overset{\varphi}{\Rightarrow} & \\ {\scriptstyle q}\searrow & & \swarrow{\scriptstyle r} \\ & B & \end{array} \qquad \text{or} \qquad \begin{array}{ccc} G & \xrightarrow{\langle \hat{\varphi}, f \rangle} & B^{\mathbb{I}} \times_B H \\ {\scriptstyle q}\searrow & & \swarrow{\scriptstyle s\pi_1} \\ & B & \end{array}$$

and 2-cells $\theta\colon (f, \varphi) \longrightarrow (g, \psi)$ are 2-cells $\theta\colon f \longrightarrow g$ such that

$$\begin{array}{ccc} & q & \\ {\scriptstyle \varphi}\swarrow & & \searrow{\scriptstyle \psi} \\ rf & \xrightarrow[r\theta]{} & rg \end{array}$$

Pseudo-Exponentiability in 2-Kleisli Categories

An object Y is *pseudo-exponentiable* in a 2-category $\mathcal{K}$ if, for every Z, there is an object Z^Y and natural equivalences

$$\mathcal{K}(X \times Y, Z) \simeq \mathcal{K}(X, Z^Y)$$

Theorem (N, 2007) Suppose that $\mathcal{K}$ is a 2-category with finite products and T, η, μ is a 2-monad on $\mathcal{K}$ such that $\eta T \cong T\eta$ and $T(X \times TY) \to TX \times TY$ is an isomorphism in $\mathcal{K}$, for all X, Y. If TY is 2-exponentiable in $\mathcal{K}$, then $T(TZ^{TY}) \longrightarrow TZ^{TY}$ is an equivalence and Y is pseudo-exponentiable in the Kleisli 2-category $\mathcal{K}_T$.

Remarks

1. $|\mathcal{K}_\mathbf{T}| = |\mathcal{K}|$, $\mathcal{K}_\mathbf{T}(X, Y) = \mathcal{K}(X, TY)$, with composition via μ.
2. One can show that the 2-monad on $\mathbf{Gpd}(\mathcal{C})/B$ related to $B^{\mathbb{I}}$ satisfies the hypotheses of the theorem.

Pseudo-Exponentiability in $\mathbf{Gpd}(\mathcal{C})//B$

Corollary $G \longrightarrow B$ is pseudo-exponentiable in $\mathbf{Gpd}(\mathcal{C})//B$, if the components of $B^{\mathbb{I}} \times_B G \xrightarrow{s\pi_1} B$ are exponentiable in $\mathcal{C}$.

Corollary $\mathbf{Gpd}(\mathcal{C})//B$ is pseudo-cartesian closed, if the diagonals $B_i \xrightarrow{\Delta_i} B_i \times B_i$ are exponentiable in a cartesian closed category $\mathcal{C}$.

Note So, $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{M}})//B$ is pseudo-cartesian closed, for $\mathcal{M} = \mathcal{K}$ or $\mathcal{E}$, if the diagonals $B_i \xrightarrow{\Delta_i} B_i \hat{\times} B_i$ are locally closed.

Corollary If $\mathcal{C}$ is locally cartesian closed category, then $\mathbf{Gpd}(\mathcal{C})$ is pseudo-locally cartesian closed.

Note A proof that $\mathbf{Gpd}(\mathbf{Sets})$ is pseudo-locally cartesian closed appeared in a 2003 paper by Palmgren posted on the arxiv.