Hypercategories

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Introduction

- Preliminary report on **Cat**-indexed categories, a.k.a. hypercategories
- More questions than answers
- Four parts
  1. Indexed categories
  2. Double categories
  3. Families of categories
  4. Derivators

I. Indexed Categories

Genesis

- Grew out of the master plan of developing mathematics based on an arbitrary elementary topos $\mathbf{S}$ (rather than a fixed set theory, ZFC e.g.)
- Mathematics is best done using category theory
  - Small category is a category object in $\mathbf{S}$
  - Large category? E.g. $Gr(\mathbf{S})$
- Idea is that a large category should come equipped with a notion of family of objects parametrized by an object of $\mathbf{S}$

History

- 1973 Lawvere - Perugia notes - “Theory of categories over a base topos”
- 1974 Bénabou - Lectures University of Montreal - “Fibrations”
- 1974 Penon - Comptes Rendus - “Catégories localement internes”
- 1978 Paré - Schumacher - SLN - “Indexed categories”

Idea of families indexed by some structured object

- $\approx$ 1850 - Riemann - Moduli spaces
- $\approx$ 1960 - Mumford et al

Definition

- **S** a category of parameters - has finite limits
- **S**-indexed category $\mathscr{A}$
  - For every I in **S** a category $\mathscr{A}(I)$ of *I-indexed families*: $\langle A_i \rangle_{i \in I}$
  - For every $\alpha : J \longrightarrow I$ a functor $\alpha^* : \mathscr{A}(I) \longrightarrow \mathscr{A}(J)$, the *reindexing*: $\alpha^* \langle A_i \rangle_{i \in J} = \langle A_{\alpha j} \rangle_{j \in J}$
  - Natural isomorphisms

$$\phi_I : 1_{\mathscr{A}(I)} \xrightarrow{\cong} 1_I^*$$

$$\phi_{\alpha,\beta} : \beta^* \alpha^* \xrightarrow{\cong} (\alpha\beta)^*$$

  - Satisfying coherence conditions: two unit triangles and one associativity pentagon
- $\mathscr{A}(\ ) : \mathbf{I}^{op} \longrightarrow \underline{\underline{CAT}}$ is a pseudo-functor
- If ϕ_I, $\phi_{\alpha,\beta}$ are identities, we say $\mathscr{A}$ is *rigid*

Examples

- $\mathbf{S} = \mathbf{Set}$, $\mathbf{A}$ arbitrary category

$$\mathscr{A}(I) = \mathbf{A}^I = \prod_I \mathbf{A}$$

  - Object is an actual I-family $\langle A_i \rangle_{i \in I}$ of $\mathbf{A}$ objects
  - Morphism is $\langle f_i \rangle : \langle A_i \rangle \longrightarrow \langle B_i \rangle$ an I-family of morphisms $f_i : A_i \longrightarrow B_i$
  - $\alpha^* \langle A_i \rangle_{i \in I} = \langle A_{\alpha j} \rangle_{j \in J}$ reindexing

- $\mathbf{S}$ an elementary topos

  - $\mathcal{S}(I) = \mathbf{S}/I$, $\alpha^* : \mathbf{S}/I \longrightarrow \mathbf{S}/J$ is pullback

    $\mathcal{S}$ is $\mathbf{S}$ indexed by itself

  - Groups in $\mathbf{S}$ are indexed by

$$Gr(\mathcal{S})(I) = Gr(\mathbf{S}/I)$$

    α^* preserves groups

Morphisms

- An *indexed functor* $F : \mathscr{A} \longrightarrow \mathscr{B}$ is a pseudo-natural transformation
  - For every I, $F(I) : \mathscr{A}(I) \longrightarrow \mathscr{B}(I)$ a functor
  - For every α, a natural isomorphism

$$\begin{tikzcd} \mathscr{A}(I) \arrow[r, "F(I)"] \arrow[d, "\alpha^*"'] \arrow[dr, phantom, "\psi_\alpha \swarrow"] & \mathscr{B}(I) \arrow[d, "\alpha^*"] \\ \mathscr{A}(J) \arrow[r, "F(J)"'] & \mathscr{B}(J) \end{tikzcd}$$

  - Satisfying "obvious" coherence conditions
- It is *rigid* if all ψ_α are identities
- An *indexed natural transformation* $t : F \longrightarrow G$ is a modification
  - For every I, a natural transformation $t(I) : F(I) \longrightarrow G(I)$
  - Compatible with the ψ_α
- Get a 2-category **S**-$\underline{\underline{\mathit{IndCat}}}$

Externalization

- Let $\mathbb{C} = (C_2 \overset{p_1}{\underset{p_2}{\overset{m}{\rightrightarrows}}} C_1 \overset{d_0}{\underset{d_1}{\overset{\mathrm{id}}{\leftrightarrows}}} C_0)$ be a category object in $\mathbf{S}$

  The *externalization* of $\mathbb{C}$ is the indexed category $\mathscr{E}x(\mathbb{C})$

  - $\mathscr{E}x(\mathbb{C})(I)$ - Objects $I \longrightarrow C_0$

    - Morphisms $I \longrightarrow C_1$

  - $\alpha : J \longrightarrow I$, $\alpha^* : \mathscr{E}x(\mathbb{C})(I) \longrightarrow \mathscr{E}x(\mathbb{C})(J)$

$$(I \longrightarrow C_0) \longmapsto (J \xrightarrow{\alpha} I \longrightarrow C_0)$$

- $\mathscr{E}x(\mathbb{C})$ is a rigid $\mathbf{S}$-indexed category

- Also have $\mathscr{E}x(F)$, $\mathscr{E}x(t)$ for internal functors and natural transformations

- $\underline{\underline{Cat}}(\mathbf{S}) \xrightarrow{\mathscr{E}x} \underline{\underline{Rigid\text{-}\mathbf{S}\text{-}IndCat}}$ is 2-full and faithful

Smallness

- $\mathscr{A}$ is *small* if it is isomorphic to $\mathscr{E}x(\mathbb{C})$ for some category object $\mathbb{C}$ in $\mathbf{S}$

- $\mathscr{A}$ has *small homs* if for every I and A, $B \in \mathscr{A}(I)$ there exist $\hom(A, B) : H(A, B) \longrightarrow I$ and a natural bijection

$$\frac{\begin{tikzcd} J \arrow[rr] \arrow[dr, "\alpha"'] & & H(A,B) \arrow[dl, "hom(A,B)"] \\ & I & \end{tikzcd} \quad \text{in} \quad \mathbf{S}/I}{\alpha^* A \longrightarrow \alpha^* B \quad \text{in} \quad \mathscr{A}(J)}$$

- When $\mathbf{S} = \mathbf{Set}$, $\mathbf{S}/I \simeq \mathbf{Set}^I$, $\mathscr{A}(I) = \mathbf{A}^I$

$$\hom(A, B) = \langle \mathbf{A}(A_i, B_i) \rangle_{i \in I}$$

- If $\mathscr{A}$ has small homs, $\mathscr{A}(I)$ enriched in $\mathbf{S}/I$ (with cartesian product)

Σ Π

- $\mathscr{A}$ has *indexed sums* if
  - for every $\alpha : J \longrightarrow I$, $\alpha^* : \mathscr{A}(I) \longrightarrow \mathscr{A}(J)$ has a left adjoint
$$\sum_\alpha : \mathscr{A}(J) \longrightarrow \mathscr{A}(I)$$
  - (*Beck condition*) for every pullback
$$\begin{tikzcd} L \arrow[r, "\beta"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "\gamma"] \\ J \arrow[r, "\alpha"'] & I \end{tikzcd}$$
    the canonical morphism
$$\begin{tikzcd} \mathscr{A}(L) \arrow[r, "\sum_\beta"] & \mathscr{A}(K) \\ \mathscr{A}(J) \arrow[u, "\delta^*"] \arrow[r, "\sum_\alpha"'] \arrow[ur, phantom, "\Longrightarrow"] & \mathscr{A}(I) \arrow[u, "\gamma^*"'] \end{tikzcd}$$
    is invertible

- Beck condition says that $\sum_\alpha$ is "pointwise"
- For $\mathbf{S} = \mathbf{Set}$

$$\sum_\alpha (\langle A_j \rangle_{j \in J}) = \langle \sum_{\alpha(j)=i} A_j \rangle_{i \in I}$$

- $\prod_\alpha$ is dual: right adjoint to α^* satisfying a similar Beck condition
- For $\mathcal{A} = \mathcal{S}$, $\sum_\alpha$, $\prod_\alpha$ are well-known from topos theory

  $\sum$ exists for any $\mathbf{S}$ and $\prod$ exists if and only if $\mathbf{S}$ is locally cartesian-closed

Remarks

- Once the basic position of equipping a category with a notion of abstract families is taken, the general theory is easy, especially for one with a basic acquaintance with topos theory
- The more like sets $\mathbf{S}$ is, the more like ordinary category theory $\mathbf{S}$-indexed categories will be. E.g. $\mathbf{S}$ a topos or spaces of some sort
- The basic ideas are
  - natural notion of family parametrized by objects of $\mathbf{S}$
  - a notion of sums and products of these families
  - naturally occurring category objects in $\mathbf{S}$
- The families intuition can be very useful

Categories as Parametrizers

- **Cat** is a good candidate for parameters
  - Almost a topos: cartesian-closed, sums are disjoint and universal
  - Well, certainly very "space like"
  - Families indexed by categories - fibrations, diagrams, ...
  - Kan extensions are sums or products of such families
  - Category objects in **Cat** are double categories
- Basic theory is straightforward - concentrate on the differences specific to our choice of **Cat** as category of parameters
  - Not locally cartesian-closed but we have a characterization of those functors for which $\prod$ exists, the Conduché fibrations
  - Quotients are not good - **Cat** is not a regular category

Hypercategories

Definition

A *hypercategory* is a **Cat**-indexed category

A *hyperfunctor* is a **Cat**-indexed functor

A *hypernatural transformation* is a **Cat**-indexed natural transformation

"Hypercategory" is the old Eilenberg-Kelly name for 2-category

II. Double Categories

Double Categories

- Ehresmann $\approx$ 1960
  Double category has objects, two kinds of arrows, horizontal and vertical, and cells

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "w", "\bullet" marking] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

- Horizontal and vertical compositions, giving two category structures, related by interchange
- Category object in **Cat**

$$\mathbb{A} = \mathbf{A}_2 \underset{p_2}{\overset{p_1}{\underset{m}{\rightrightarrows}}} \mathbf{A}_1 \underset{d_1}{\overset{d_0}{\underset{\mathrm{id}}{\leftarrow}}} \mathbf{A}_0$$

$\mathbf{A}_0$: objects and vertical arrows
$\mathbf{A}_1$: horizontal arrows and cells

Double Functors, Horizontal Transformations

- A *double functor* $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a function taking objects, horizontal (resp. vertical) arrows, and cells of $\mathbb{A}$ to similar elements of $\mathbb{B}$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "v_1"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & A_2 \arrow[d, "v_2"] \\
A_3 \arrow[r, "f_3"'] & A_4
\end{tikzcd}
\quad \overset{F}{\longmapsto} \quad
\begin{tikzcd}
FA_1 \arrow[r, "Ff_1"] \arrow[d, "Fv_1"'] \arrow[dr, phantom, "\overset{F\alpha}{\Rightarrow}"] & FA_2 \arrow[d, "Fv_2"] \\
FA_3 \arrow[r, "Ff_2"'] & FA_4
\end{tikzcd}
$$

  preserving everything

- A *horizontal transformation* $t : F \longrightarrow G$ is
  - for every A in $\mathbb{A}$, a horizontal arrow $tA : FA \longrightarrow GA$ in $\mathbb{B}$
  - for every vertical arrow v in $\mathbb{A}$, a cell

$$
\begin{tikzcd}
FA \arrow[r, "tA"] \arrow[d, "Fv"'] \arrow[dr, phantom, "\overset{tv}{\Rightarrow}"] & GA \arrow[d, "Gv"] \\
FA' \arrow[r, "tA'"'] & GA'
\end{tikzcd}
$$

  - horizontally natural
  - vertically functorial

Examples

- Sets, functions, relations, inclusion

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "R"', "\bullet" marking] \arrow[dr, phantom, "\subseteq"] & B \arrow[d, "S", "\bullet" marking] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

- For any category $\mathbf{A}$, $\square\mathbf{A}$ *squares* in $\mathbf{A}$
- For any 2-category $\underline{\underline{A}}$, $\mathbb{Q}\underline{\underline{A}}$ *quintets* in $\underline{\underline{A}}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

- For any 2-category $\underline{\underline{A}}$, $\mathbb{H}\mathrm{or}\underline{\underline{A}}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, equal] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

Externalization

- Every double category $\mathbb{A}$ gives a hypercategory $\mathscr{E}x(\mathbb{A})$
    - Objects of $\mathscr{E}x(\mathbb{A})(\mathbf{I})$ are vertical diagrams of shape $\mathbf{I}$ in $\mathbb{A}$

$$\mathbf{I} \longrightarrow \mathbf{A_0} \quad \text{i.e.} \quad \mathbb{V}\mathrm{ert}\ \mathbf{I} \longrightarrow \mathbb{A}$$

    - Morphisms of $\mathscr{E}x(\mathbb{A})(\mathbf{I})$ are

$$\mathbf{I} \longrightarrow \mathbf{A_1}$$

      i.e., horizontal transformations of vertical diagrams

    - For $F : \mathbf{J} \longrightarrow \mathbf{I}$, $F^* : \mathscr{E}x(\mathbb{A})(\mathbf{I}) \longrightarrow \mathscr{E}x(\mathbb{A})(\mathbf{J})$ is given by composition

$$(\mathbb{V}\mathrm{ert}\mathbf{I} \xrightarrow{\Phi} \mathbb{A}) \stackrel{F^*}{\longmapsto} (\mathbb{V}\mathrm{ert}\mathbf{J} \xrightarrow{\mathbb{V}\mathrm{ert}F} \mathbb{V}\mathrm{ert}\mathbf{I} \xrightarrow{\Phi} \mathbb{A})$$

- $\mathscr{E}x(\mathbb{A})(\mathbb{1})$ has
    - objects: the objects of $\mathbb{A}$
    - arrows: the horizontal arrows of $\mathbb{A}$
- $\mathscr{E}x(\mathbb{A})$ is a rigid hypercategory

Examples

- $\mathscr{E}x(\Box\mathbf{A})(\mathbf{I}) = \mathbf{A}^{\mathbf{I}}, \quad F^* = \mathbf{A}^F : \mathbf{A}^{\mathbf{I}} \longrightarrow \mathbf{A}^{\mathbf{J}}$

  $\prod_F$, $\sum_F$ are Kan extensions *but* Beck condition doesn't hold!

- $\mathscr{E}x(\mathbb{Q}\underline{\underline{A}})(\mathbf{I})$
  - objects are 2-functors $\mathbf{I} \longrightarrow \underline{\underline{A}}$
  - morphisms are lax transformations

- $\mathscr{E}x(\mathbb{H}\mathrm{or}\underline{\underline{A}})(\mathbf{I})$
  - objects are constant on components of $\mathbf{I}$, i.e. a function $\pi_0\mathbf{I} \longrightarrow \mathrm{Ob}\ \underline{\underline{A}}$
  - morphisms are not constant
  - e.g. if $\mathbf{I}$ is connected, an object of $\mathscr{E}x(\mathbb{H}\mathrm{or}\underline{\underline{A}})(\mathbf{I})$ is an object of $\underline{\underline{A}}$, and a morphism $A \longrightarrow B$ is a functor $\mathbf{I} \longrightarrow \underline{\underline{A}}(A, B)$

Small homs

- Every small category, i.e. $\mathscr{E}x(\mathbb{A})$, has small homs
- Let $\Phi, \Psi : \mathbb{V}\mathrm{ert}\, \mathbf{I} \longrightarrow \mathbb{A}$ be two $\mathbf{I}$-families
  hom(Φ, Ψ) is given by the pullback

$$\begin{tikzcd} H(\Phi,\Psi) \arrow[r] \arrow[d, "\mathrm{hom}(\Phi,\Psi)"'] & \mathbf{A}_1 \arrow[d, "\langle d_0, d_1 \rangle"] \\ \mathbf{I} \arrow[r, "\langle \Phi, \Psi \rangle"'] & \mathbf{A}_0 \times \mathbf{A}_0 \end{tikzcd}$$

  - an object of $H(\Phi, \Psi)$ is a pair $(I, \Phi I \xrightarrow{f} \Psi I)$
  - a morphism is a pair (i, ϕ)

$$\begin{tikzcd} I \arrow[d, "i"'] & \Phi I \arrow[r, "f"] \arrow[d, "\Phi_i"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & \Psi I \arrow[d, "\Psi_i"] \\ J & \Phi J \arrow[r, "g"'] & \Psi J \end{tikzcd}$$

Remark

hom(Φ, Ψ) *is a family of categories, an object of* $\mathbf{Cat}/\mathbf{I}$ *(general theory). It can be an* arbitrary *category over* $\mathbf{I}$, $\mathbf{X} \longrightarrow \mathbf{I}$ *(take* $\mathbf{I} + \mathbf{X} + \mathbf{X} + \mathbf{I} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathbf{I} + \mathbf{X} + \mathbf{I} \;\substack{\longrightarrow\\ \longrightarrow}\; \mathbf{I} + \mathbf{I}$)

Tabulators

- For $\mathscr{A} = \mathscr{E}x\mathbb{A}$
$$\prod_{\mathbb{2}} : \mathscr{A}(\mathbb{2}) \longrightarrow \mathscr{A}(\mathbb{1})$$
gives tabulators for $\mathbb{A}$

- The adjointness $\mathbb{2}^* \longrightarrow\!\!\!\!\!\dashv \prod_{\mathbb{2}}$ gives the 1-dimensional universal property

$$\begin{tikzcd} & A \arrow[dd, "\bullet" marking, "v"] \\ X \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Rightarrow"] & {} \\ & B \end{tikzcd} \quad \Bigg| \quad X \to T(v)$$

Tetrahedron

- Beck for

$$
\begin{tikzcd}
\mathbb{2} \times \mathbb{2} \arrow[r, "P_2"] \arrow[d, "P_1"'] & \mathbb{2} \arrow[d, "\mathbb{2}"] \\
\mathbb{2} \arrow[r, "\mathbb{2}"'] & \mathbb{1}
\end{tikzcd}
$$

gives the tetrahedron condition

$$
\begin{tikzcd}
X \arrow[dd, "z"', "\bullet" marking] \arrow[dr] \arrow[drr, bend left] \arrow[dr, phantom, shift left=3ex, "\Rightarrow"] & & \\
\arrow[r, phantom, "\Rightarrow"] & A \arrow[r, "\bullet" marking] & B \\
Y \arrow[ur] \arrow[urr, bend right] \arrow[ur, phantom, shift right=3ex, "\Rightarrow"] & &
\end{tikzcd}
\qquad \Bigg| \qquad
\begin{tikzcd}
X \arrow[dd, "z"', "\bullet" marking] \arrow[dr] & \\
\arrow[r, phantom, "\Rightarrow"] & T(v) \\
Y \arrow[ur] &
\end{tikzcd}
$$

2-Functoriality

- $\mathscr{E}x(\mathbb{A})(-) : \mathbf{Cat}^{op} \longrightarrow \mathbf{CAT}$ is a functor but not usually a 2-functor!
-
$$\begin{tikzcd} \mathbf{J} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "t\Downarrow"] & \mathbf{I} \end{tikzcd} \qquad \mapsto \qquad \begin{tikzcd} \mathscr{E}x(\mathbb{A})(\mathbf{I}) \arrow[r, bend left, "F^*"] \arrow[r, bend right, "G^*"'] \arrow[r, phantom, "?"] & \mathscr{E}x(\mathbb{A})(\mathbf{J}) \end{tikzcd}$$

  Given an $\mathbf{I}$-family $\Phi : \mathbf{I} \longrightarrow \mathbf{A}_0$ we get

$$\begin{tikzcd} \Phi F \arrow[r, "\Phi t"] \arrow[d, equal] & \Phi G \arrow[d, equal] \\ F^*\Phi \arrow[r] & G^*\Psi \end{tikzcd}$$

  But this is a vertical transformation ($\Phi t(J)$ is a vertical arrow)
  A natural transformation $F^* \longrightarrow G^*$ is a horizontal transformation
- $\mathscr{E}x(\Box\mathbf{A})$ is 2-functorial
- $\mathscr{E}x(\mathbb{Q}\underline{\underline{A}})$ is not in general

Weak Double Categories

- Most double categories of structures are weak
- A *weak* double category has the same data as a double category except that vertical composition is only unitary and associative up to coherent special isomorphism, e.g.

$$
\begin{tikzcd}
A_0 \arrow[r, equal] \arrow[dd, "v_2 \cdot v_1"'] & A_0 \arrow[d, "v_1"] \\
 & A_1 \arrow[dd, "v_3 \cdot v_2"] \\
A_2 \arrow[d, "v_3"'] \arrow[r, phantom, "\overset{\kappa}{\Longrightarrow}"] & \\
A_3 \arrow[r, equal] & A_3
\end{tikzcd}
$$

- It can be viewed as a weak category object

$$
\begin{tikzcd}
\mathbf{A}_3 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathbf{A}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbf{A}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbf{A}_0 \arrow[l]
\end{tikzcd}
$$

in $\underline{\underline{CAT}}$

$\mathbb{C}\mathrm{at}$

- There is a weak double category $\mathbb{C}\mathrm{at}$ that plays the role of **Cat** in the double category universe
- It consists of small categories, functors, profunctors, natural transformations

$$\begin{tikzcd} \mathbf{A} \arrow[r, "F"] \arrow[d, "P"', "\bullet" marking] \arrow[dr, phantom, "\overset{t}{\Rightarrow}"] & \mathbf{C} \arrow[d, "Q", "\bullet" marking] \\ \mathbf{B} \arrow[r, "G"'] & \mathbf{D} \end{tikzcd}$$

$$t : P(-,-) \longrightarrow Q(F-, G-)$$

- Profunctors are "relations between categories"
- $P : \mathbf{A} \longrightarrow\!\!\!\!\!\!\!\bullet\;\; \mathbf{B}$ is $P : \mathbf{A}^{op} \times \mathbf{B} \longrightarrow \mathbf{Set}$

Other Examples

- $\mathbb{S}\mathrm{et}$: sets, functions, spans
- $\mathbb{R}\mathrm{ing}$: rings, homomorphisms, bimodules
- $\mathbb{V}\mathrm{ert}\underline{\underline{B}}$: $\underline{\underline{B}}$ a bicategory
- $\mathbf{V}$-$\mathbb{C}\mathrm{at}$: suitable $\mathbf{V}$
- $\mathbf{V}$-$\mathbb{S}\mathrm{et}$: sets, functions, $\mathbf{V}$-matrices

The Hypercategory of a Weak Double Category

- $\mathscr{H}yp(\mathbb{A})(\mathbf{I})$
  - objects pseudo-functors $\mathbb{V}\mathrm{ert}\ \mathbf{I} \longrightarrow \mathbb{A}$
  - morphisms are pseudo-natural transformations
  - F^* is precomposition with F
- $\mathscr{H}yp(\mathbb{A})$ is a rigid hypercategory
- $\mathscr{H}yp(\mathbb{A})$ has small homs
- The natural notion of morphism is pseudo-functor (or weak functor) $\Phi : \mathbb{A} \longrightarrow \mathbb{B}$
- This gives $\mathscr{H}yp(\mathbb{A}) \xrightarrow{\mathscr{H}yp(\Phi)} \mathscr{H}yp(\mathbb{B})$ a rigid hyperfunctor

Proposition

There is an equivalence of categories

$$\frac{\mathbb{A} \xrightarrow{\textit{weak}} \mathbb{B}}{\mathscr{H}yp(\mathbb{A}) \xrightarrow{\textit{rigid}} \mathscr{H}yp(\mathbb{B})}$$

Essential Smallness

- A category is essentially small if it is equivalent to a small one
- Equivalence for indexed categories (Bunge-Paré, Cahiers 1980)
    - Strong equivalence $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ such that $FG \cong 1$, $GF \cong 1$
    - Weak equivalence
        - F full and faithful ($F(I)$ is full and faithful for every I)
        - F essentially surjective on objects: for every I, $B \in \mathscr{B}(I)$ there is a cover $e : J \twoheadrightarrow I$, $A \in \mathscr{A}(J)$, and an isomorphism $e^*B \cong F(J)(A)$
- There *cover* meant regular epimorphism, and **S** was assumed to be regular

Effective Descent Morphisms

- **Cat** is not regular
- A good notion of cover is an effective descent morphism

Theorem

(Janelidze, Sobral, Tholen) A functor $F : \mathbf{J} \longrightarrow \mathbf{I}$ *is of effective descent type in* **Cat** *if and only if it is onto on objects, arrows, and composable pairs of objects*

Theorem

If $\mathbb{A}$ *is a weak double category, then there exist a strict double category* $\mathbb{B}$ *and a weak equivalence* $\mathscr{E}x(\mathbb{B}) \longrightarrow \mathscr{H}yp(\mathbb{A})$

III. Families of Categories

The Indexing of *Cat*

- There are many possible notions of what a family of categories or functors might be
  - $\mathbf{Cat}/\mathbf{I}$ categories over $\mathbf{I}$ with commutative triangles
  - $\mathbf{Cat}/\!\!/\mathbf{I}$ categories over $\mathbf{I}$ with lax triangles
  - $\mathbf{Fib}/\mathbf{I}$ fibrations over $\mathbf{I}$ with cartesian functors
  - $\mathbf{Cond}/\mathbf{I}$ Conduché fibrations with commutative triangles
  - $\mathrm{Pseudo}(\mathbf{I}^{op}, \mathbf{Cat})$
  - Etc.

The Standard Indexing

- $\mathscr{C}at(\mathbf{I}) = \mathbf{Cat}/\mathbf{I}$, F^* is pullback along F
- $\mathscr{C}at(\mathbb{1}) \cong \mathbf{Cat}$
- $\mathscr{C}at(\mathbb{2})$ has objects profunctors $\mathbf{A} \longrightarrow\!\!\!\!\!\!\bullet\;\; \mathbf{B}$
  morphisms natural transformations

$$\begin{tikzcd} \mathbf{A} \arrow[r] \arrow[d, "\bullet" marking] & \mathbf{C} \arrow[d, "\bullet" marking] \\ \mathbf{B} \arrow[r] \arrow[ur, phantom, "\Rightarrow"] & \mathbf{D} \end{tikzcd}$$

- $\mathscr{C}at$ is a flexible hypercategory

Lax Functors

Theorem

(Bénabou) A category over $\mathbf{I}$, $U : \mathbf{A} \longrightarrow \mathbf{I}$ *is equivalent to a normal lax functor* $\mathbf{I} \longrightarrow \mathrm{Prof}$

- In fact $\mathbf{Cat}/\mathbf{I}$ is equivalent to the category of normal lax functors $\mathbb{V}\mathrm{ert}\ \mathbf{I} \longrightarrow \mathbb{C}\mathrm{at}$ with horizontal transformations

- The correspondence is given by

  $I \longmapsto \mathbf{A}_I = U^{-1}(I)$

  $i \longmapsto P_i$

  $P_I(A, A') = \{a : A \longrightarrow A' | Ua = i\}$

- This does not use choice

- If instead we take this as our definition of $\mathscr{C}at$, then we get an equivalent but rigid hypercategory

Some Properties of $\mathscr{C}at$

- $\mathscr{C}at$ has $\sum_F$ satisfying the Beck condition (true in general for any $\mathbf{S}$)
- $\mathscr{C}at(\ ) : \mathbf{Cat}^{op} \longrightarrow \mathbf{CAT}$ is not a 2-functor
  - Let $I, I' : \mathbb{1} \longrightarrow \mathbf{I}$ be two functors and $i : I \longrightarrow I'$ a natural transformation. Then we have

$$\begin{tikzcd} \mathscr{C}at(\mathbf{I}) \arrow[r, shift left, "I^*"] \arrow[r, shift right, "I'^*"'] & \mathscr{C}at(\mathbb{1}) \end{tikzcd}$$

$$\text{i.e} \quad \begin{tikzcd} \mathbf{Cat}/I \arrow[r, shift left] \arrow[r, shift right] & \mathbf{Cat} \end{tikzcd}$$

$$\begin{array}{ccc} \mathbf{A} & \xmapsto{I^*} & \mathbf{A}_I \\ \downarrow & & \\ \mathbf{I} & \xmapsto[I'^*]{} & \mathbf{A}_{I'} \end{array}$$

but there may not be any functor $\mathbf{A}_I \longrightarrow \mathbf{A}_{I'}$ (if $\mathbf{A}_I \neq \emptyset = \mathbf{A}_{I'}$)

Powerful Families

- An object A in a cartesian category $\mathbf{A}$ is *powerful* if $A \times (\ ) : \mathbf{A} \longrightarrow \mathbf{A}$ has a right adjoint $(\ )^A$

Theorem

(Giraud, Conduché) $U : \mathbf{A} \longrightarrow \mathbf{I}$ is powerful in $\mathbf{Cat}/\mathbf{I}$ if and only if it satisfies the following condition: for every $f : A \longrightarrow A'$, every factorization of Uf, $UA \xrightarrow{x} I \xrightarrow{y} UA'$ lifts to a factorization of f, $A \xrightarrow{g} \bar{A} \xrightarrow{h} A'$, $Ug = x$, $Uh = y$, and any two liftings are connected by a zigzag path of diagrams

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, equal] & \bar{A} \arrow[r, "h"] \arrow[d, "a"] & A' \arrow[d, equal] \\ A \arrow[r, "g'"'] & \bar{A}' \arrow[r, "h'"'] & A' \end{tikzcd}$$

with $Ua = 1_I$

Powerful Categories

Proposition

(1) *Powerful morphisms are stable under pullback*

(2) $F^* : \mathbf{Cat}/\mathbf{I} \longrightarrow \mathbf{Cat}/\mathbf{J}$ *has a right adjoint* $\prod_F$ *if and only if* F *is powerful*

(3) $\prod_F$ *satisfies the Beck condition*

- Get a subhypercategory $\mathscr{PCat}$ of $\mathscr{Cat}$ with $\mathscr{PCat}(\mathbf{I})$ the full subcategory of $\mathbf{Cat}/\mathbf{I}$ determined by the powerful families
- $\mathscr{PCat}$ has small homs
- $\mathbf{A} \longrightarrow \mathbf{I}$ is powerful if and only if the corresponding lax functor $\mathbb{V}\mathrm{ert}\ \mathbf{I} \longrightarrow \mathbb{C}\mathrm{at}$ is in fact pseudo
- $\mathscr{PCat} \cong \mathscr{Hyp}(\mathbb{C}\mathrm{at})$

Conduché at Work

- Useful to see in detail how the Conduché condition comes in
  - Let $U : \mathbf{A} \longrightarrow \mathbf{I}$ and $V : \mathbf{B} \longrightarrow \mathbf{I}$ be such that V^U exists in $\mathbf{Cat}/\mathbf{I}$. Say $V^U : \mathbf{C} \longrightarrow \mathbf{I}$
  - An object of $\mathbf{C}$ has to be $(I, \Phi : \mathbf{A}_I \longrightarrow \mathbf{B}_I)$
    A morphism of $\mathbf{C}$ has to be

$$\begin{tikzcd}
I \arrow[d, "i"'] & \mathbf{A}_I \arrow[r, "\Phi"] \arrow[d, "P_i"'] & \mathbf{B}_I \arrow[d, "Q_i"] \\
I' & \mathbf{A}_{I'} \arrow[r, "\Phi'"'] \arrow[ur, phantom, "\overset{\phi}{\Rightarrow}"] & \mathbf{B}_{I'}
\end{tikzcd}$$

  - Composition

$$\begin{tikzcd}
I \arrow[d, "i"'] & \mathbf{A}_I \arrow[r, equal] \arrow[dd, "P_{i'i}"'] & \mathbf{A}_I \arrow[r, "\Phi'"] \arrow[d, "P_i"'] & \mathbf{B}_I \arrow[r, equal] \arrow[d, "Q_I"] & \mathbf{B}_I \arrow[dd, "Q_{i'i}"] \\
I' \arrow[d, "i'"'] & \arrow[r, phantom, "\overset{\mu^{-1}_{i',i}}{\Rightarrow}"] & \mathbf{A}_{I'} \arrow[r, "\Phi'"] \arrow[d, "P_{i'}"'] \arrow[ur, phantom, "\overset{\phi}{\Rightarrow}"] & \mathbf{B}_{I'} \arrow[d, "Q_i"] \arrow[r, phantom, "\overset{\nu_{i',i}}{\Rightarrow}"] & {} \\
I'' & \mathbf{A}_{I''} \arrow[r, equal] & \mathbf{A}_{I''} \arrow[r, "\Phi''"'] \arrow[ur, phantom, "\overset{\phi'}{\Rightarrow}"] & \mathbf{B}_{I''} \arrow[r, equal] & \mathbf{B}_{I''}
\end{tikzcd}$$

Contrafamilies

Definition

An $\mathbf{I}$-indexed *contrafamily* of categories is a normal oplax functor $\mathbb{V}\mathrm{ert}\mathbf{I} \longrightarrow \mathbb{C}\mathrm{at}$
A morphism of contrafamilies is a horizontal transformation

- If Φ is a contrafamily and Ψ a family, the pointwise exponential $\Psi^{\Phi}(I) = \Psi(I)^{\Phi(I)}$ is a family
- Precomposition gives reindexing functors for a hypercat $\mathcal{C}on\mathcal{C}at$
- Note that $\mathcal{C}on\mathcal{C}at(\mathbb{1}) \cong \mathbf{Cat}$

Theorem

$\mathcal{C}on\mathcal{C}at(\mathbf{I})$ *is cartesian closed*

Measuring

- Let Φ be a contrafamily and Ψ, Θ families. A Φ-*measuring* from Ψ to Θ is a morphism of families $\Psi \longrightarrow \Theta^{\Phi}$

  - $$M(I) : \Phi I \times \Psi I \longrightarrow \Theta I$$

  - $$\begin{tikzcd} \Phi I \times \Psi I \arrow[r, "MI"] \arrow[d, "\Phi(i)\times\Psi(i)"'] \arrow[dr, phantom, "\overset{Mi}{\Rightarrow}"] & \Theta I \arrow[d, "\Theta i"] \\ \Phi I' \times \Psi I' \arrow[r, "MI'"'] & \Theta I' \end{tikzcd}$$

  - Compatible with the laxity and colaxity cells

Theorem

Given two families Ψ, Θ, there is a universal measuring from Ψ to Θ

$$M(\Psi, \Theta) \times \Psi \longrightarrow \Theta$$

Corollary

$\mathscr{C}at(\mathbf{I})$ *is enriched in* $\mathscr{C}on\mathscr{C}at(\mathbf{I})$ *and is cotensored*

IV. Derivators

Derivators

- Heller - Homotopy theories (1988)
- Grothendieck - Les Dérivateurs (1990)
- Franke - System of triangulated diagram categories (1996?)

Definition

A *derivator* is

- (Der 0) A 2-functor (strict) $\mathscr{A} : \mathbf{Cat}^{op} \longrightarrow \mathbf{CAT}$
- (Der 1) $\mathscr{A}(\sum \mathbf{I}_\alpha) \cong \prod \mathscr{A}(\mathbf{I}_\alpha)$
- (Der 2) $\mathscr{A}(\mathbf{I}) \longrightarrow \prod_{\mathbf{I}} \mathscr{A}(\mathbb{1})$ conservative
- (Der 3) F^* has a left adjoint $\sum_F$ and a right adjoint $\prod_F$
- (Der 4) Beck-Chevalley for comma categories

$$\begin{tikzcd} \mathbf{L} \arrow[r, "G"] \arrow[d, "\Psi"'] \arrow[dr, phantom, "\swarrow"] & \mathbf{K} \arrow[d, "\Phi"] \\ \mathbf{J} \arrow[r, "F"'] & \mathbf{I} \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \mathscr{A}(\mathbf{L}) \arrow[r, "\sum_G"] \arrow[dr, phantom, "\searrow\cong"] & \mathscr{A}(\mathbf{K}) \\ \mathscr{A}(\mathbf{J}) \arrow[u, "\Psi^*"] \arrow[r, "\sum_F"'] & \mathscr{A}(\mathbf{I}) \arrow[u, "\Phi^*"'] \end{tikzcd} \quad \begin{tikzcd} \mathscr{A}(\mathbf{L}) \arrow[d, "\prod_\Psi"'] \arrow[dr, phantom, "\cong\swarrow"] & \mathscr{A}(\mathbf{K}) \arrow[l, "G^*"'] \arrow[d, "\prod_\Phi"] \\ \mathscr{A}(\mathbf{J}) & \mathscr{A}(\mathbf{I}) \arrow[l, "F^*"] \end{tikzcd}$$

- (Der 5) The canonical $\mathscr{A}(\mathbb{2} \times \mathbf{I}) \longrightarrow \mathscr{A}(\mathbf{I})^{\mathbb{2}}$ is essentially surjective on objects and full

Derivators as Hypercategories

- What do (Der 0)-(Der 5) mean in terms of hypercategories? In particular for $\mathscr{E}x(\mathbb{A})$?

Der 0

- (Der 0): $\mathscr{A}(-) : \mathbf{Cat}^{op} \longrightarrow \mathbf{CAT}$ is a 2-functor

  - For $\mathbb{1} \underset{1}{\overset{0}{\underset{\leftarrow !}{\rightrightarrows}}} \mathbb{2}$ we have $0 \dashv ! \dashv 1$ so $\mathscr{A}(\mathbb{2}) \underset{1^*}{\overset{0^*}{\underset{\leftarrow !^*}{\rightrightarrows}}} \mathscr{A}(\mathbb{1})$, $1^* \dashv !^* \dashv 0^*$

    For $\mathscr{A} = \mathscr{E}x(\mathbb{A})$
    $\mathscr{E}x(\mathbb{A})(\mathbb{2})$ is the category whose objects are vertical arrows and whose morphisms are cells
    $\mathscr{E}x(\mathbb{A})(\mathbb{1})$ is the category whose objects are those of $\mathbb{A}$ and whose morphisms are horizontal arrows
    0^* is the domain functor, 1^* is the codomain functor and $!^*$ is the functor "identity"

$$
\begin{tikzcd}
A \arrow[d, "v"'] & A \\
B & B
\end{tikzcd}
\quad
\begin{tikzcd}
{} \arrow[r, mapsto, "\mathrm{dom}"] & {} \\
{} \arrow[r, mapsto, "\mathrm{cod}"'] & {}
\end{tikzcd}
\qquad\qquad
A \xmapsto{\mathrm{id}}
\begin{tikzcd}
A \arrow[d, "\mathrm{id}"] \\
A
\end{tikzcd}
$$

$$\mathrm{cod} \dashv \mathrm{id} \dashv \mathrm{dom}$$

cod ⊣ id ⊣ dom

- (dom)(id) $\cong$ 1 so id is full and faithful

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \\ A \arrow[r, "g"'] & B \end{tikzcd} \qquad \Rightarrow \qquad f = g \;\&\; \alpha = \mathrm{id}_f$$

- cod ⊣ id

$$\begin{tikzcd} A \arrow[d, "v"', "\bullet" marking] & \\ B \arrow[r, "g"'] & C \end{tikzcd} \quad \exists! \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & C \arrow[d, "\mathrm{id}_C", "\bullet" marking] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

- id ⊣ dom

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d, "w", "\bullet" marking] \\ & C \end{tikzcd} \quad \exists! \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & B \arrow[d, "w", "\bullet" marking] \\ A \arrow[r, "g"'] & C \end{tikzcd}$$

Companions

- This means that every vertical arrow v has a horizontal companion $v_\circ$
- Furthermore $(\ )_\circ$ is functorial
- Finally

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & C \arrow[d, "w"] \\ B \arrow[r, "g"'] & D \end{tikzcd} \quad \cong \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"'] & C \arrow[r, "w_\circ"] \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & D \arrow[d, "\mathrm{id}_D"] \\ A \arrow[r, "v_\circ"'] & B \arrow[r, "g"'] & D \end{tikzcd}$$

- I.e., for any boundary $(f, g; v, w)$ there is at most one cell and there is one if and only if $w_\circ f = sv_\circ$

Lax Kernel

- Let **A**, **B** be two categories with the same objects and $\Phi : \mathbf{A} \longrightarrow \mathbf{B}$ a functor, identity on objects. We can construct a double category $\mathbb{K}_\Phi$ with objects those of **A** (and/or **B**), vertical arrows the morphisms of **A** and horizontal arrows the morphisms of **B**. There is a unique cell

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & C \arrow[d, "a'"] \\ B \arrow[r, "b'"'] & D \end{tikzcd}$$

iff $(\Phi a')b = b'(\Phi a)$

- $\mathbb{K}_\Phi$ is a kind of lax kernel of Φ

$$(\Phi, \Phi, \Phi) \rightrightarrows\!\!\!\to (\Phi, \Phi) \rightrightarrows \mathbf{A} \xrightarrow{\Phi} \mathbf{B}$$

Proposition

$\mathscr{E}x(\mathbb{A})$ *satisfies (Der 0) if and only if* $\mathbb{A}$ *is of the form* $\mathbb{K}_\Phi$ *for some* Φ

Der 1

- (Der 1) $\mathscr{A}(\sum \mathbf{I}_\alpha) \cong \prod \mathscr{A}(\mathbf{I}_\alpha)$
- This says that a family indexed by a sum of categories is an ordinary family of families indexed by each component
    - $\mathscr{E}x(\mathbb{A})$ always satisfies this
    - So does $\mathscr{H}yp(\mathbb{A})$
    - Also $\mathscr{C}at, \mathscr{P}\mathscr{C}at, \mathscr{C}on\mathscr{C}at$

Der 2

- (Der 2) $\mathscr{A}(\mathbf{I}) \longrightarrow \prod_{\mathbf{I}} \mathscr{A}(\mathbb{1})$ conservative

- When $\mathscr{A} = \mathscr{E}x(\mathbb{A})$, this says that if f and g in

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & C \arrow[d, "w"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

are invertible, then so is α

  - E.g.: For a bicategory $\underline{\underline{B}}$, $\mathbb{V}\mathrm{ert}\underline{\underline{B}}$ satisfies this if and only if $\underline{\underline{B}}$ is groupoid enriched
  - For $\mathscr{E}x(\mathbb{A})$ this follows from (Der 0)
  - For $\mathscr{E}x(\mathbb{A})$ and $\mathscr{H}yp(\mathbb{A})$ we do have

  a weaker condition

$$\mathscr{A}(\mathbf{I}) \longrightarrow \prod_{Arr\mathbf{I}} \mathscr{A}(\mathbb{2})$$

  is conservative

Der 3

- (Der 3) F^* has a left adjoint $\sum_F$ and a right adjoint $\prod_F$

  These are the usual adjoints from indexed category theory, though they are usually required to satisfy the Beck condition for pullbacks

  - For $\mathscr{A} = \mathscr{E}x(\mathbb{A})$ or $\mathscr{H}yp(\mathbb{A})$ and $\mathbf{I} : \mathbf{I} \longrightarrow \mathbb{1}$, $\prod_{\mathbf{I}} : \mathscr{A}(\mathbf{I}) \longrightarrow \mathscr{A}(\mathbb{1})$ gives the horizontal limit of a vertical diagram, like tabulators e.g.

  - For general $F : \mathbf{I} \longrightarrow \mathbf{J}$, $\prod_F : \mathscr{A}(\mathbf{I}) \longrightarrow \mathscr{A}(\mathbf{J})$ is a kind of horizontal Kan extension

Der 4

- (Der 4) The Beck condition for comma objects requires $\mathscr{A}(\ )$ to be a 2-functor in order to get the comparisons

$$\begin{tikzcd} \mathscr{A}(\mathbf{L}) \arrow[r, "\sum_G"] \arrow[dr, phantom, "\searrow"] & \mathscr{A}(\mathbf{K}) \\ \mathscr{A}(\mathbf{J}) \arrow[u, "\Psi^*"] \arrow[r, "\sum_F"'] & \mathscr{A}(\mathbf{I}) \arrow[u, "\Phi^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{A}(\mathbf{L}) \arrow[d, "\prod_\Psi"'] & \mathscr{A}(\mathbf{K}) \arrow[l, "G^*"'] \arrow[d, "\prod_\Phi"] \arrow[dl, phantom, "\swarrow"] \\ \mathscr{A}(\mathbf{J}) & \mathscr{A}(\mathbf{I}) \arrow[l, "F^*"] \end{tikzcd}$$

Der 5

- (Der 5) $H_{\mathbf{I}} : \mathscr{A}(\mathbb{2} \times \mathbf{I}) \longrightarrow \mathscr{A}(\mathbf{I})^{\mathbb{2}}$ essentially surjective on objects and full
- Also depends on $\mathscr{A}(\ )$ being a 2-functor
- In fact $H_{\mathbf{I}}$ is the embodiment of 2-functoriality
  - Given $\begin{tikzcd} \mathbf{I} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "t\Downarrow"] & \mathbf{J} \end{tikzcd}$ we get $\mathbb{2} \times \mathbf{I} \xrightarrow{T} \mathbf{J}$

    so $\mathscr{A}(\mathbf{J}) \xrightarrow{T^*} \mathscr{A}(\mathbb{2} \times \mathbf{I})$

    If we compose with $H_{\mathbf{I}}$ we get $\mathscr{A}(\mathbf{J}) \xrightarrow{H_{\mathbf{I}} T^*} \mathscr{A}(\mathbf{I})^{\mathbb{2}}$

    Thus $t^* : F^* \longrightarrow G^*$

H as a Hyperfunctor

- We can define new hypercategories from old as follows
  - $\mathscr{A}[\mathbb{2}]$ is given by $\mathscr{A}[\mathbb{2}](\mathbf{I}) = \mathscr{A}(\mathbb{2} \times \mathbf{I})$

    This is the hypercategory of internal (or vertical) arrows of $\mathscr{A}$
  - $\mathscr{A}^{\mathbb{2}}$ is given by $\mathscr{A}^{\mathbb{2}}(\mathbf{I}) = \mathscr{A}(\mathbf{I})^{\mathbb{2}}$

    This is the hypercategory of external (or horizontal) arrows of $\mathscr{A}$
- $H : \mathscr{A}[\mathbb{2}] \longrightarrow \mathscr{A}^{\mathbb{2}}$ is a hyperfunctor

  It is an assignment of horizontal arrows to vertical ones
- For a double category $\mathbb{A}$

  $\mathscr{E}x(\mathbb{A})[\mathbb{2}]$ corresponds to $\mathbb{A}^{\mathbb{V}\mathrm{ert}\,\mathbb{2}}$

  $\mathscr{E}x(\mathbb{A})^{\mathbb{2}}$ corresponds to $\mathbb{A}^{\mathbb{H}\mathrm{or}\,\mathbb{2}}$

  $H : \mathbb{A}^{\mathbb{V}\mathrm{ert}\,\mathbb{2}} \longrightarrow \mathbb{A}^{\mathbb{H}\mathrm{or}\,\mathbb{2}}$

H for $\mathscr{E}x(\mathbb{K}_\Phi)$

- H for $\mathscr{E}x(\mathbb{K}_\Phi)$ is always full and faithful

- For it to be essentially surjective on objects as in (Der 5) means that there is a functor S and an isomorphism

$$\begin{tikzcd} \mathbf{B}^{\mathbb{2}} \arrow[r, "S"] \arrow[dr, "1_{\mathbf{B}^{\mathbb{2}}}"'] & \mathbf{A}^{\mathbb{2}} \arrow[d, "\Phi^{\mathbb{2}}"] \arrow[dl, phantom, "\cong"] \\ & \mathbf{B}^{\mathbb{2}} \end{tikzcd}$$

To be continued ...