CT2017 - UBC - Vancouver

Quasi-toposes as elementary quotient completions

Fabio Pasquali
University of Padova

j.w.w.

M. E. Maietti and G. Rosolini

Elementary doctrines

An elementary doctrine is a functor

$$P\colon \mathcal{C}^{op} \to \mathcal{InfSL}$$

such that

- $\mathcal{C}$ has finite products
- reindexing of the form

  $$P(id_X \times \Delta_A)\colon P(X \times A \times A) \to P(X \times A)$$

  have a left adjoint

  $$\exists_{id_X \times \Delta_A}\colon P(X \times A) \to P(X \times A \times A)$$

- $\exists_{id_X \times \Delta_A}(\alpha) = P(\langle \pi_1, \pi_2 \rangle)(\alpha) \wedge P(\langle \pi_2, \pi_3 \rangle)(\delta_A)$

where $\delta_A = \exists_{\Delta_A}(\top_A)$ is the equality predicate over A

[F.W. Lawvere. Equality in hyperdoctrines and comprehension scheme as an adjoint functor. 1970]

Elementary doctrines (examples)

Subsets. $\mathcal{P}\colon \mathcal{S}et^{op} \to \mathcal{I}nf\mathcal{SL}$

$$\delta_A = \{(x, y) \in A \mid x = y\}$$

Subobjects. $\mathcal{C}$ has finite limits. **Sub**: $\mathcal{C}^{op} \to \mathcal{I}nf\mathcal{SL}$

$$\delta_A \text{ is } \Delta_A\colon A \to A \times A$$

Weak subobjects. $\mathcal{C}$ has finite limits. $\Psi\colon \mathcal{C}^{op} \to \mathcal{I}nf\mathcal{SL}$ is

$$A \mapsto (\mathcal{C}/A)_{\mathrm{po}}$$

Pullbacks gives $\Psi(f)$

$$\delta_A \text{ is } \Delta_A\colon A \to A \times A$$

Strong equality

An elementary doctrine $P\colon \mathcal{C}^{op} \to \mathcal{InfSL}$ has strong equality if for every pair of arrows $f, g\colon X \to Y$ it is

$$f = g \quad \text{If and only if} \quad \top_X = P(\langle f, g\rangle)(\delta_Y)$$

Equivalence relations

$P \colon \mathcal{C}^{op} \to \mathcal{InfSL}$ is an elementary doctrine

ρ in $P(A \times A)$ is a P-equivalence relation over A if ρ is

reflexive: $\delta_A \leq \rho$

symmetric: $P(\langle \pi_2, \pi_1 \rangle)(\rho) \leq \rho$

transitive: $P(\langle \pi_1, \pi_2 \rangle)(\rho) \wedge P(\langle \pi_2, \pi_3 \rangle)(\rho) \leq P(\langle \pi_1, \pi_3 \rangle)(\rho)$

Effective quotients

An elementary doctrine $P\colon \mathcal{C}^{op} \to \mathcal{InfSL}$ has quotients if for every A in $\mathcal{C}$ and every P-equivalence relation ρ over A there is an arrow

$$q\colon A \to A/\rho$$

such that

$$\rho \leq P(q \times q)(\delta_{A/\rho})$$

and for every $f\colon A \to Y$ with $\rho \leq P(f \times f)(\delta_Y)$ there is a unique $k\colon A/\rho \to Y$ with $kq = f$.

Quotients are said effective when $\rho = P(q \times q)(\delta_{A/\rho})$

Elementary quotient completion

Suppose $P\colon \mathcal{C}^{op} \to \mathcal{InfSL}$ is an elementary doctrine. Consider the category $\mathcal{Q}_P$ where

objects: (A, ρ) where ρ is a a P-equivalence relation over A

arrows: $[f]\colon (A, \rho) \longrightarrow (B, \sigma)$ where $f\colon A \to B$ is in $\mathbb{C}$ such that $\rho \leq P(f \times f)(\sigma)$ and $g \in [f]$ if and only if $\top_A \leq P(\langle f, g\rangle)(\sigma)$

Elementary quotient completion

Consider the functor

$$
\begin{array}{ccc}
(A,\rho) & & \{\phi \in P(A) \mid P(\pi_1)(\phi) \wedge \rho \le P(\pi_2)(\phi)\} \\
{\scriptstyle [f]}\big\downarrow & \mapsto & \big\uparrow {\scriptstyle P(f)} \\
(B,\sigma) & & \{\phi \in P(B) \mid P(\pi_1)(\phi) \wedge \sigma \le P(\pi_2)(\phi)\}
\end{array}
$$

$P_q\colon \mathcal{Q}_P^{op} \longrightarrow \mathcal{InfSL}$ is the elementary quotient completion of $P\colon \mathcal{C}^{op} \to \mathcal{InfSL}$

If σ is a P_q-equivalence relation over (A,ρ), the quotient is

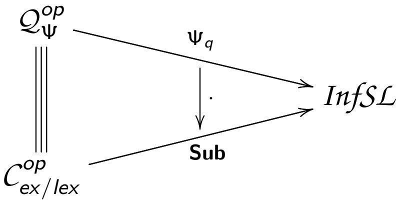
$$[id_A]\colon (A,\rho) \to (A,\sigma)$$

[M. E. Maietti, G. Rosolini. Elementary quotient completion. 2013]

Example: the ex/lex completion

$\mathcal{C}$ has finite limits

$\Psi\colon \mathcal{C}^{op} \longrightarrow \mathcal{InfSL}$ is the doctrine of weak subobjects



Projectivity

An object X of $\mathcal{C}$ is said q-projective if for every diagram of the form

$$\begin{array}{ccc} & & Y \\ & & \big\downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y/\rho \end{array}$$

there is an arrow $k\colon X \to Y$ such that

$$\top_X = P(\langle sk, f \rangle)(\delta_{Y/\rho})$$

Enough q-projective

An elementary doctrine $P\colon \mathcal{C}^{op} \to \mathit{InfSL}$ is said to have enough q-projectives if every object is the effective quotient of a q-projective object.

Enough q-projective

Theorem: An elementary doctrine $P\colon \mathcal{C}^{op} \to \mathcal{InfSL}$ with effective quotients and strong equality is of the form $P'_q\colon \mathcal{Q}_{P'}^{op} \to \mathcal{InfSL}$ for some $P'\colon \mathcal{C}'^{op} \to \mathcal{InfSL}$ if and only it has enough q-projectives and these are closed under binary products.

First order doctrine

A first order doctrine is an elementary doctrine $P\colon \mathcal{C}^{op} \to \mathcal{InfSL}$ where

i) $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Heyt}$

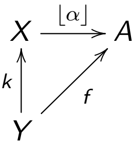
ii) for every projection $\pi_A\colon A \times X \to A$, the map $P(\pi_A)$ has both a right adjoint ($\forall_{\pi_A}$) and a left adjoint ($\exists_{\pi_A}$) natural in A (Beck-Chevalley condition)

Weak comprehension

An elementary doctrine $P\colon \mathcal{C}^{op} \to \mathcal{InfSL}$ has weak comprehensions if for every A in $\mathcal{C}$ and every α in $P(A)$, there is an arrow

$$\lfloor\alpha\rfloor\colon X \longrightarrow A$$

with $P(\lfloor\alpha\rfloor)(\alpha) = \top_X$ such that for every arrow $f\colon Y \longrightarrow A$ with $P(f)(\alpha) = \top_Y$ there is $k\colon Y \longrightarrow X$ making commute



Comprehension is full if $P(\lfloor\alpha\rfloor)(\alpha) \leq P(\lfloor\alpha\rfloor)(\beta)$ iff $\alpha \leq \beta$

The doctrine of weak subobjects has full weak comprehension

Properties of the elementary quotient completion

Suppose $P\colon \mathcal{C}^{op} \to \mathcal{I}nf\mathcal{SL}$ is a first order doctrine with weak full comprehensions and strong equality.

Theorem: $\mathcal{Q}_P$ has finite limits

Theorem: $\mathcal{C}$ has weak U iff $\mathcal{Q}_P$ has U, where U is any of

- finite coproducts
- natural number object
- parametrized list objects
- arbitrary limits (if arbitrary meets in the fibers)
- arbitrary coproducts (if arbitrary joins in the fibers)
- a classifier of comprehensions

Theorem: $\mathcal{C}$ is weak U iff $\mathcal{Q}_P$ is U, where U is any of

- cartesian closed
- locally cartesian closed

Application: the ex/lex completion

$\mathcal{C}$ has finite limits

$\Psi \colon \mathcal{C}^{op} \longrightarrow \mathcal{InfSL}$ is the doctrine of weak subobjects

Theorem (Carboni-Rosolini): $\mathcal{C}_{ex/lex}$ is lcc iff $\mathcal{C}$ is weakly lcc.

Theorem (Menni): $\mathcal{C}_{ex/lex}$ is an elementary topos iff $\mathcal{C}$ is weakly locally cartesian closed with a weak proof classifier.

Triposes

A tripos is a first order doctrine with weak powerobjects

i.e. for every A in $\mathcal{C}$ there is $\mathbb{P}A$ in $\mathcal{C}$ and $\in_A$ in $P(A \times \mathbb{P}A)$ such that for every ψ in $P(A \times Y)$ there is $\{\psi\}: Y \to \mathbb{P}A$ such that $P(id_A \times \{\psi\})(\in_A) = \psi$

Every tripos $P: \mathcal{C}^{op} \to \mathcal{InfSL}$ canonically generates an elementary topos $\mathcal{C}[P]$ via the Tripos-To-Topos construction.

[J. M. E. Hyland, P. T. Johnstone, A. M. Pitts. Tripos theory. 1980]
[A. M. Pitts. Tripos theory in retrospect. 2002]

Quasitoposes

A quasitopos is a finitely complete, finitely cocomplete, locally cartesian closed category in which there exists an object that classifies strong monomorphisms

An arithmetic quasitopos a quasitopos with a NNO

Quasitoposes

Theorem: If $P\colon \mathcal{C}^{op} \to \mathit{InfSL}$ is a tripos with weak full comprehension, where $\mathcal{C}$ is weakly locally cartesian closed, with weak co-products and a weak natural number objects, then $\mathcal{Q}_P$ is an arithmetic quasitopos.

Quasitoposes

Theorem: If $P\colon \mathcal{C}^{op} \to \mathit{InfSL}$ is a tripos with weak full comprehension, where $\mathcal{C}$ is weakly locally cartesian closed, with weak co-products and a weak natural number objects, then $\mathcal{Q}_P$ is an arithmetic quasitopos.

Remark: NNO + lcc give list objects.

List objects give the transitive closure of a relation

The coequalizer of $f, g\colon A \to B$ is the quotient of the equivalence relation over B generated by

$$\exists a\ (f(a) = b \wedge g(a) = b')$$

Applications

We have already commented on the ex/lex completion of a category with finite limits.

We shall discuss also

General equilogical spaces

Assemblies

Bishop total setoids model over CIC

Applications: General equilogical spaces

$\mathcal{P}\colon \mathcal{T}op^{op} \longrightarrow \mathcal{I}nf\mathcal{SL}$ maps a space A to the powerset of its set of points and each continuous functions to the inverse image mapping.

$\mathcal{P}$ is a tripos: $\mathbb{P}(A)$ is $\{0,1\}^A$ and $\in_A\colon A \times \{0,1\}^A \to \{0,1\}$

$\mathcal{P}$ has full comprehensions: subspaces

$\mathcal{T}op$ is weakly locally cartesian closed with a natural number object ($\mathbb{N}$ discrete)

Each $\mathcal{P}(A)$ has arbitrary meets and joins and these are preserved by maps of the form $\mathcal{P}(f)$

$\mathcal{Q}_\mathcal{P}$ is $\mathcal{G}equ$.

Corollary: $\mathcal{G}equ$ is an arithmetic quasi-topos which is complete and cocomplete.

Applications: Assemblies

Denote by $\mathcal{Asm}$ the quasitopos of assemblies.

S-Sub: $\mathcal{Asm}^{op} \longrightarrow \mathcal{InfSL}$ is the tripos of strong subobjects

This tripos has effective quotients and strong equality.

$\mathcal{Asm}$ has enough q-projectives and these are the partitioned assemblies.

Then **S-Sub** is the elementary quotient completion of the restriction of **S-Sub** to $\mathcal{PAsm}$

Applications: Calculus of Inductive Constructions (CIC)

Denote by $\mathcal{CT}$ the category whose objects are closed types of CIC and an arrow $A \to B$ is an equivalence class of terms $t\colon B[x\colon A]$ where t and t' are equivalent if there is $p\colon Id_B(t,t')[x\colon A]$

$Pr(A)$ denotes the poset reflection of the order whose elements are propositions depending on A where $B \leq C$ if $q\colon B \Rightarrow C[x\colon A]$. The action of Pr on arrows of $\mathcal{CT}$ is given by substitution.

The pair $(\mathcal{CT}, Pr)$ is a tripos with weak full comprehension.

$\mathcal{CT}$ is weakly lcc with a weak NNO.

Q_{Pr} is equivalent to the setoid model.

Corollary: The total setoid model over CIC is an arithmetic quasitopos

Conclusions

$P\colon \mathcal{C}^{op} \to \mathcal{InfSL}$	$\mathcal{Q}_P$	$\mathcal{C}[P]$
$\mathcal{P}\colon \mathcal{Top}^{op} \to \mathcal{InfSL}$	$\mathcal{Gequ}$	$\mathcal{Set}$
$\mathbf{S\text{-}Sub}\colon \mathcal{PAsm}^{op} \to \mathcal{InfSL}$	$\mathcal{Asm}$	$\mathcal{Set}$

Conclusions

$P\colon \mathcal{C}^{op} \to \mathcal{InfSL}$	$\mathcal{Q}_P$	$\mathcal{C}[P]$
$\mathcal{P}\colon \mathcal{Top}^{op} \to \mathcal{InfSL}$	$\mathcal{Gequ}$	$\mathcal{Set}$
$\textbf{S-Sub}\colon \mathcal{PAsm}^{op} \to \mathcal{InfSL}$	$\mathcal{Asm}$	$\mathcal{Set}$

$\mathcal{Q}_P \equiv \mathcal{C}[P]$ iff the tripos P_q validates AUC

Study of models of type theories that do not validate AUC, such as CIC (Coquand, Paulin-Mohring) or the Minimalist Foundation (Maietti, Sambin)

Thank you