The Orbifold Construction for Join Restriction Categories

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Outline

Restriction Categories

A **restriction category** is a category equipped with a *restriction combinator*

$$\frac{f : A \to B}{\overline{f} : A \to A}$$

which satisfies:

[R1] $\overline{f}f = f$

[R2] $\overline{f}\overline{g} = \overline{g}\overline{f}$

[R3] $\overline{f}\overline{g} = \overline{\overline{f}g}$

[R4] $f\overline{g} = \overline{fg}f$

Restriction Categories - Some Basic Facts and Concepts

- A map f is *total* when $\overline{f} = 1$.
- Total maps form a subcategory of any restriction category.
- $\overline{\overline{f}} = \overline{f}$ and we refer to maps e with $e = \overline{e}$ as *restriction idempotents*.
- The *restriction ordering* on maps is given by,

$$f \leq g \text{ if and only if } \overline{f}g = f.$$

This makes a restriction category poset-enriched.

Joins

Definition

- Two parallel maps f and g in a restriction category are **compatible**, written $f \smile g$, when $\bar{f}g = \bar{g}f$.
- A restriction category is a **join restriction category** when for each compatible set of maps S the join

$$\bigvee_{s \in S} s$$

exists and is preserved by composition in the sense that

$$f(\bigvee_{s \in S} s)g = \bigvee_{s \in S}(fsg).$$

The Manifold Construction - Objects

The manifold construction as first introduced by Grandis, and then reformulated by Cockett and Cruttwell:

Definition

An **atlas** in a join restriction category $\mathbb{B}$ consists of a family of objects $(X_i)_{i \in I}$ of $\mathbb{B}$, with, for each $i, j \in I$, a map $\phi_{ij} \colon X_i \to X_j$ such that for each $i, j, k \in I$,

[Atl.1] $\phi_{ii}\phi_{ij} = \phi_{ij}$ (partial charts);

[Atl.2] $\phi_{ij}\phi_{jk} \leq \phi_{ik}$ (cocycle condition);

[Atl.3] ϕ_{ij} is the partial inverse of ϕ_{ji} (partial inverses).

Remark

Note that this set of data corresponds to a lax functor from the chaotic (or, indiscrete) category on I to $\mathbb{B}$.

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The Manifold Construction - Arrows

Definition

Let (X_i, ϕ_{ij}) and (Y_k, ψ_{kh}) be atlases in $\mathbb{B}$. An **atlas map** $A\colon (X_i, \phi_{ij}) \to (Y_k, \psi_{kh})$ is a familiy of maps

$$X_i \xrightarrow{A_{ik}} Y_k$$

such that

- $\phi_{ii} A_{ik} = A_{ik}$;
- $\phi_{ij} A_{jk} \leq A_{ik}$;
- $A_{ik} \psi_{kh} = \overline{A_{ik}} A_{ih}$ (the *linking condition*).

Orbifolds

- Orbifold charts are given by charts consisting of an open subset of $\mathbb{R}^n$ with an action by a finite group.
- An orbifold atlas may contain non-identity homeomorphisms from a chart to itself (induced by the group action) and parallel embeddings between two charts.
- So we want to replace the chaotic category indexing the atlas for a manifold by an inverse category.

Inverse Categories

- A map $f : A \to B$ in a restriction category is called a **restricted isomorphism**, or **partial isomorphism**, if there is a map $f^\circ : B \to A$ such that $ff^\circ = \overline{f}$ and $f^\circ f = \overline{f^\circ}$. (Restricted inverses are unique.)
- A restriction category in which all maps are restricted isomorphisms is an **inverse category**.

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Linking Functors

Definition

Let $\mathbb{X}$ and $\mathbb{Y}$ be restriction categories, a map of the underlying directed graphs $F : \mathbb{X} \to \mathbb{Y}$ is a **linking functor** when:

[LFun1] $\overline{F(x)} \leq F(\overline{x})$,

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A manifold in a restriction category $\mathbb{B}$ is given by a linking functor from a chaotic category into $\mathbb{B}$.

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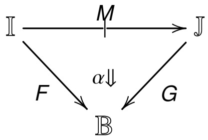
The Category Orb($\mathbb{B}$)

- The objects of Orb($\mathbb{B}$) are linking functors from inverse categories into $\mathbb{B}$,

$$F\colon \mathbb{I} \to \mathbb{B}.$$

- The arrows of Orb($\mathbb{B}$) are deterministic restriction bimodules over $\mathbb{B}$,



Restriction Bimodules

A **restriction bimodule** $\mathbb{X} \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} \mathbb{Y}$ between restriction categories $\mathbb{X}$ and $\mathbb{Y}$ consists of

- A set $M(X, Y)$ for each $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$ (for $v \in M(X, Y)$ we write $X \overset{v}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} Y$);
- Actions of the category $\mathbb{X}$ on the left and $\mathbb{Y}$ on the right, satisfying

$$\begin{array}{lll} 1 \cdot v = v & (xx') \cdot v = x \cdot (x' \cdot v) & v \cdot 1 = v \\ v \cdot (yy') = (v \cdot y) \cdot y' & (x \cdot v) \cdot y = x \cdot (v \cdot y). & \end{array}$$

- A restriction operation

$$\frac{X \overset{v}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} Y}{X \xrightarrow{\overline{v}} X}$$

  satisfying
  - $\overline{(\overline{v})} = \overline{v}$ (hence, $\overline{v}$ is a restriction idempotent in $\mathbb{X}$);
  - $\overline{v} \cdot v = v$
  - $v \cdot \overline{y} = \overline{v \cdot y} \cdot v$

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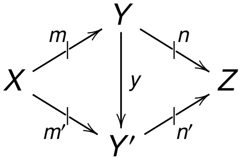
Restriction Bimodules

- Composition: for restriction bimodules $\mathbb{X} \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} \mathbb{Y} \overset{N}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} \mathbb{Z}$, composition is given by $M \otimes N$.
- An element $m \otimes n$ of $M \otimes N(X, Z)$ is given by $m \in M(X, Y)$ and $n \in N(Y, Z)$ and the equivalence relation is generated by: for



we have $m \otimes n = m \otimes y \cdot m' \sim m \cdot y \otimes n' = m' \otimes n'$.
- The restriction on this bimodule is given by

$$\overline{m \otimes n} = \overline{m \cdot \overline{n}}.$$

- The identity module $1_{\mathbb{X}}$ is given by $1_{\mathbb{X}}(X, X') = \mathbb{X}(X, X')$.

The Category of Restriction Bimodules

- For a restriction bimodule $M\colon \mathbb{X} \nrightarrow \mathbb{Y}$ the restriction bimodule $\overline{M}\colon \mathbb{X} \nrightarrow \mathbb{X}$ is given by

$$\overline{M}(X, X') = \{\overline{m}f;\, m \in M(X, Y) \text{ and } f \in \mathbb{X}(X, X')\} \subseteq \mathbb{X}(X, X').$$

- [DeWolf, 2017] The category of restriction bimodules with invertible restriction modulations is a restriction bicategory.
- Hence we obtain a restriction category when we take isomorphism classes of restriction bimodules.

Deterministic Bimodules

Definition

A restriction bimodule $\mathbb{X} \overset{M}{\rightarrow\!\!\!\!\!\!+\;} \mathbb{Y}$ is *deterministic* if for each pair of $m_1 \in M(X, Y)$ and $m_2 \in M(X, Y')$ there is an arrow $y\colon Y \to Y'$ in $\mathbb{Y}$ such that $m_1 \cdot y = \overline{m_1} \cdot m_2$,

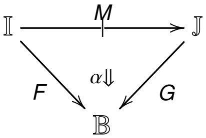
$$\begin{array}{ccc} X & \overset{m_1}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & Y \\ {\scriptstyle \overline{m_1}}\downarrow & & \downarrow{\scriptstyle y} \\ X & \underset{m_2}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & Y'. \end{array}$$

Remarks

- If $\mathbb{I}$ is an inverse category, the module $1_{\mathbb{I}}$ is deterministic, and so is each module of the form $\overline{M}$, where $M \colon \mathbb{I} \nrightarrow \mathbb{J}$.
- Deterministic modules are closed under composition.

Restriction Bimodules over $\mathbb{B}$

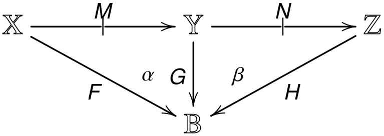
- A bimodule (profunctor) $M\colon \mathbb{X} \nrightarrow \mathbb{Y}$ between ordinary categories corresponds to a bipartite category (collage) $\mathbb{C}_M(\mathbb{X}, \mathbb{Y})$ on the disjoint union of the objects of $\mathbb{X}$ and $\mathbb{Y}$.
- When M is a restriction bimodule, $\mathbb{C}_M(\mathbb{X}, \mathbb{Y})$ is a restriction category.
- A restriction bimodule over $\mathbb{B}$,



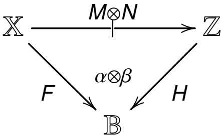
consists of a linking functor $\alpha\colon \mathbb{C}_M(\mathbb{X}, \mathbb{Y}) \to \mathbb{B}$ which restricts to F on $\mathbb{X}$ and to G on $\mathbb{Y}$.

Composition of Restriction Bimodules over $\mathbb{B}$

For deterministic restriction bimodules



we define



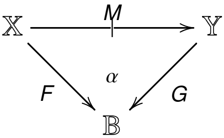
by

$$\alpha \otimes \beta(m \otimes n) = \bigvee_{(m',n')\sim(m,n)} \alpha(m')\beta(n')$$

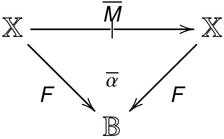
and this is again a linking functor from a deterministic restriction bimodule into $\mathbb{B}$.

The Restriction Operation

For a deterministic restriction module,



we define $\overline{\alpha}$ in



by

$$\overline{\alpha}(f) = \bigvee_{m:\ \delta_0(m)=\delta_0(f)} \overline{\alpha(m)}F(f)$$

The Restriction Category Orb($\mathbb{B}$)

Theorem

The category Orb($\mathbb{B}$) *with objects linking functors* $F : \mathbb{I} \to \mathbb{B}$ *and arrows equivalence classes of deterministic bimodules over* $\mathbb{B}$ *is a restriction category.*

Joins for Orb($\mathbb{B}$)

Let

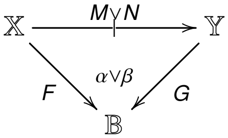
$$\begin{array}{ccc} \mathbb{X} & \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\mid\;\;\;\;\;\;\;} & \mathbb{Y} \\ & {}_F\searrow \;\; \alpha \;\; \swarrow_G & \\ & \mathbb{B} & \end{array} \qquad \begin{array}{ccc} \mathbb{X} & \overset{N}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\mid\;\;\;\;\;\;\;} & \mathbb{Y} \\ & {}_F\searrow \;\; \beta \;\; \swarrow_G & \\ & \mathbb{B} & \end{array}$$

be arrows in Orb($\mathbb{B}$) with an equivalence $\tau\colon \overline{M}\otimes N \overset{\sim}{\Rightarrow} \overline{N}\otimes M$. We define the **join** $M \vee N$ by the pushouts

$$\begin{array}{ccccc} \overline{N}\otimes M(X,Y) & \underset{\sim}{\xrightarrow{\tau(X,Y)}} & \overline{M}\otimes N(X,Y) & \xrightarrow{\cdot} & N(X,Y) \\ \Big\downarrow\cdot & & & & \Big\downarrow \\ M(X,Y) & & \longrightarrow & & M\vee N(X,Y) \end{array}$$

Joins for Orb($\mathbb{B}$)

We define $\alpha \vee \beta$ in



by

$$\alpha \vee \beta(p) = \begin{cases} \alpha(p) & \text{if } p \in M(X, Y) \\ \beta(b) & \text{if } p \in N(X, Y) \end{cases}$$

Theorem

The category Orb($\mathbb{B}$) *with objects linking functors* $F : \mathbb{I} \to \mathbb{B}$ *and arrows equivalence classes of deterministic restriction bimodules over* $\mathbb{B}$ *is a join restriction category.*

Orbifold Atlases [Joint with A. Sibih]

For an orbifold $(X, \mathcal{U})$, the atlas $\mathcal{U}$ consists of

- Charts, $(\tilde{U}_i, G_i, \rho_i, \varphi_i)$ where $\rho_i \colon G_i \to \mathrm{Homeo}(\widetilde{U}_i, \widetilde{U}_i)$ and $\varphi \colon \widetilde{U}_i \to \widetilde{U}_i/G_i \xrightarrow{\sim} U_i \subseteq X$.
- An index poset $I = \mathcal{O}(\mathcal{U}) \subseteq \mathcal{O}(X)$ such that any intersection of two atlas opens is the union of smaller atlas opens.
- Pseudofunctors and a vertical transformation,

$$\mathbb{HI} \underset{(C,\gamma)}{\overset{(A,\alpha)}{\rightrightarrows}} \mathbb{G}\mathrm{roupmod} \qquad \Downarrow\rho$$

into the double category of groups, bimodules (as horizontal arrows) and group homomorphisms (as vertical arrows).

Atlas Modules

- For each $U_i \subseteq U_j$, we obtain a double cell,

$$\begin{array}{ccc} G_i & \overset{A_{ij}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\;} & G_j \\ {\scriptstyle \rho_i}\downarrow & \rho_{ij} & \downarrow{\scriptstyle \rho_j} \\ G_i^{\text{red}} & \underset{C_{ij}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\;} & G_j^{\text{red}} \end{array}$$

- C_{ij} contains the actual embeddings between the charts.
- A_{ij} and C_{ij} are *atlas modules*: all groups act freely and the codomain groups act transitively. Furthermore, G_i acts transitively on the fibers of ρ_{ij}.

The Inverse Category for $\mathcal{U}$

The poset I together with the elements of the modules A_{ij} can be used to build a category $\mathbb{I}_{\mathcal{U}}$:

- Objects are the elements of I.
- An arrow $i \to j$ is given by an element $\lambda \in A_{ij}$.
- The composition of $i \xrightarrow{\lambda} i' \xrightarrow{\lambda'} i''$ is given by $\alpha_{ii'i''}(\lambda, \lambda') \in A_{ii''}$.
- For each $i \in I$, $\alpha_{ii} \colon A_{ii} \xrightarrow{\sim} G_i$, and we obtain the identity arrow $1_i = \alpha_{ii}(e_{G_i})$.

We turn $\mathbb{I}_{\mathcal{U}}$ into an inverse category $\widehat{\mathbb{I}_{\mathcal{U}}}$ by freely adding all partial inverses.

Orbifold Inverse Categories

The finite isotropy groups and local compatibility conditions on an orbifold atlas give us that the resulting inverse category has the following properties:

- **Orbital** For each object i there is a finite group G_i of total maps and for any other endomorphism $\theta\colon i \to i$ there is a $g_\theta \in G$ such that $\theta \leq g_\theta$.
- **Local Compatibility** For any map of the form $\theta\zeta^\circ : i \to j$ there is a finite collection of maps $\omega_k^\circ \xi_k$ such that $\theta\zeta^\circ = \bigvee_{k=1}^{n} \omega_k^\circ \xi_k$ and all ω_k and ξ_k are total.

Remark

It follows that each map in $\widehat{\mathbb{I}_{\mathcal{U}}}$ can be written as $\theta\zeta^\circ = \bigvee_{k=1}^{n} \omega_k^\circ \xi_k$ with ω_k and ξ_k in $\mathbb{I}_{\mathcal{U}}$.

The Linking Functor for $\mathcal{U}$

An orbifold $(X, \mathcal{U}, I, A, \rho, C)$ induces the linking functor:

$$\rho\colon \widehat{\mathbb{I}_{\mathcal{U}}} \to \textbf{Open}$$

- $i \mapsto \widetilde{U}_i$ on objects;
- $\lambda \mapsto \rho_{ij}(\lambda)$ for $\lambda \in A_{ij}$;
- $g \mapsto \rho_i(g)$ for $g \in G_i$;
- $\omega^\circ\xi \mapsto \rho_{ij}(\omega)^\circ\rho_{ik}(\xi)$ where $\omega \in A_{ij}$ and $\xi \in A_{ik}$;
- This can be extended to arbitrary maps, since compatible families in $\widehat{I_{\mathcal{U}}}$ are sent to compatible families in **Open**.

Maps Between Orbifold Atlases

With Sibih we developed a notion of map between orbifolds that corresponds to generalized maps between orbifold groupoids. We took our inspiration from the manifold construction.

- In the manifold construction an atlas map $\varphi\colon \mathcal{U} \to \mathcal{V}$ is given by a family of partial maps $\varphi_{ij}\colon U_i \to V_j$ between charts.
- For orbifolds, we only use partial maps that are defined on special subsets of the charts: translation subsets.
- Hence, we give a poset of such maps for each pair (i, j).

Translation Subsets of Charts

Definition

Let $\widetilde{U}$ be a chart with structure group G. A subset $V \subseteq \widetilde{U}$ is a **translation subset** if for each $g \in G_U$, either $g \cdot V = V$ or $g \cdot V \cap V = \emptyset$.

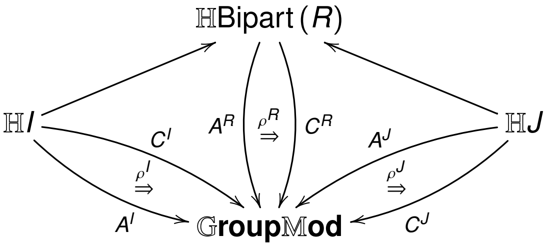
Maps Between Orbifolds

A map

$$(k; R, A^R, C^R, \rho^R) \colon \mathfrak{A} = (X; \mathcal{U}, I, A^I, C^I, \rho^I) \to \mathfrak{B} = (Y; \mathcal{V}, J, A^J, C^J, \rho^J)$$

consists of

- a continuous function $k \colon X \to Y$;
- a poset-valued profunctor $R \colon I \to J$ over $\mathbb{G}$**roup**$\mathbb{M}$**od**,



We can work with Set-valued profunctors, because the restriction will give the poset structure.

Maps Between Orbifolds, Cont'd

The data in $(k; R, A^R, C^R, \rho^R)$ need to satisfy the following conditions:

- $C^R_{ij}(r)$ is a set of functions $f\colon U_f \to \widetilde{V}_j$ with $U_f \subseteq \widetilde{U}_i$ a translation subset, and G_i^{red} and H_j^{red} act by composition. We may need to add additional structure related to this.
- For $f \in C^R_{ij}(r)$,
$$\begin{array}{ccc} U_f & \xrightarrow{f} & \widetilde{V}_j \\ {\scriptstyle \pi_i|_{U_f}}\downarrow & = & \downarrow{\scriptstyle \pi_j} \\ U_i & \xrightarrow[k|_{U_i}]{} & V_j \end{array}$$
- The U_f cover the preimage of $\widetilde{V}_j$ in $\widetilde{U}_i$:
$$\bigcup_{r \in R(i,j), f \in C_{ij}(r)} U_f = \pi_i^{-1}(k|_{U_i}^{-1}(V_j)).$$

Instead of referring back to the quotient space, we use the linking condition.

Horizontal Maps, Cont'd (action conditions)

The data in
$(k; R, A^R, C^R, \rho^R) \colon \mathfrak{A} = (X; \mathcal{U}, I, A^I, C^I, \rho^I) \to \mathfrak{B} = (Y; \mathcal{V}, J, A^J, C^J, \rho^J)$
need to satisfy the following conditions:

- H_j (resp. H_j^{red}) acts freely on $A_{ij}^R(r)$ (resp. $C_{ij}^R(r)$) for each $r \in R(i,j)$. We need to add this.
- G_i and H_j act jointly transitively on $A_{ij}^R(r)$: for $f, f' \in C_{ij}^R(r)$ there are $g \in G_i$ and $h \in H_j$ such that $h \cdot f \cdot g = f'$. This works automatically in the new set-up.
- If $f, f' \in A_{ij}^R(r)$ with $U_{\rho(f)} = U_{\rho(f')}$ then there is an $h \in H_j$ such that $f = h \cdot f'$. This is part of the deterministic condition on modules.
- If $f \in A_{ij}^R(r)$, $f' \in A_{ij}^R(r')$ with $x \in U_{\rho(f)} \cap U_{\rho(f')} \subset \widetilde{U}_i$ then there is an $s \in R(i,j)$ with $s \leq r$, $s \leq r'$ and there is an $f'' \in A_{ij}^R(s)$ with $x \in U_{f''} \subseteq U_f \cap U_{f'}$. Automatic in the restriction category set-up.

The Correspondence for Maps

- Maps in Orb(**Open**) with some additional properties give rise to orbifold atlas maps.
- (Classical) orbifold maps give rise to restriction modules over **Open**, but in general they are only locally deterministic.

Other Results and Future Work

- For classical orbifolds one tends to take larger equivalence classes of maps than we took here; there is a natural way to do this here as well, but in order to obtain joins we can only require modules to be locally deterministic;
- We would like Orb to monadic, but it isn't in the current set-up. We plan to fix this by extending the definition of Orb for to join restriction bicategories of a particular type (and let the result be a join restriction bicategory as well). We plan to show that this construction is 2-monadic.