Functoriality and topos representation for quantales of étale-covered groupoids

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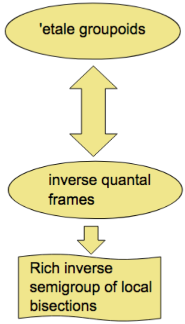
Motivation


'etale groupoids
inverse quantal frames
Rich inverse semigroup of local bisections

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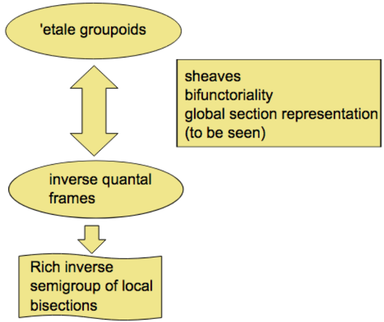
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global section representation
(to be seen)
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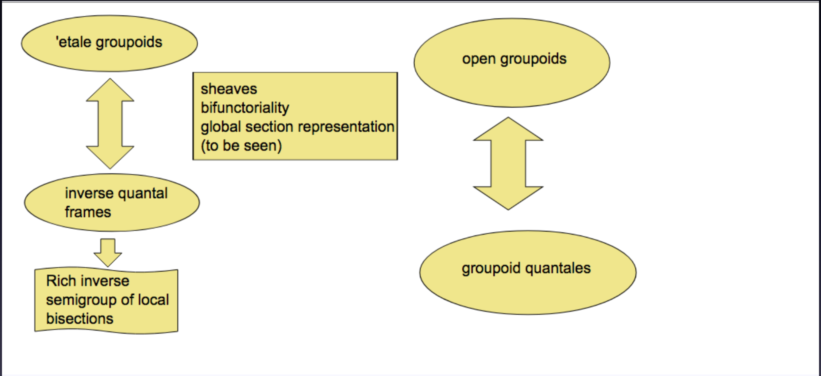
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Rich inverse semigroup of local bisections
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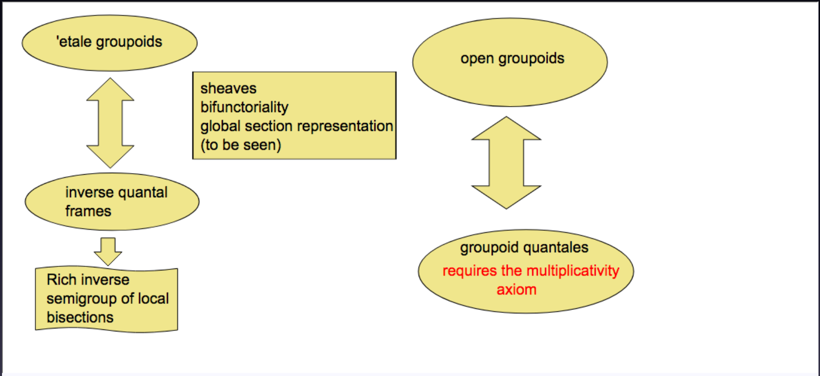

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requires the multiplicativity axiom

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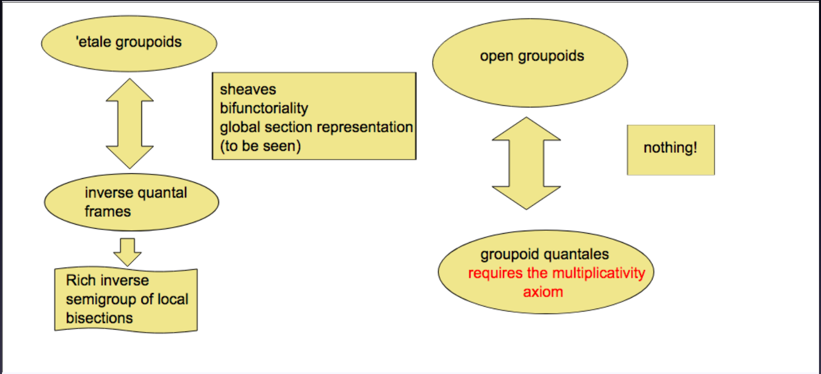
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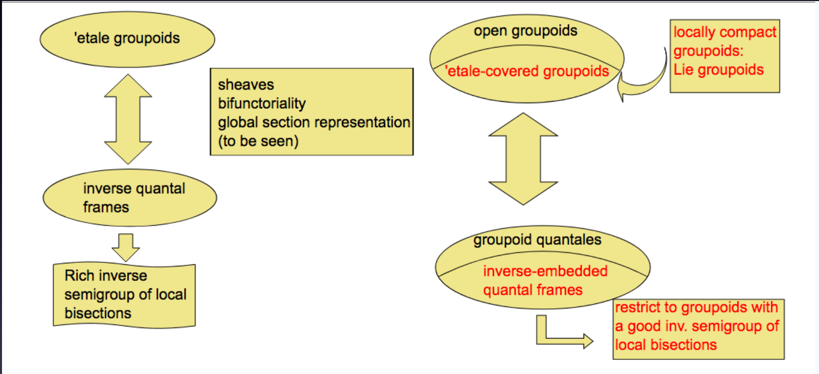
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Lie groupoids
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restrict to groupoids with a good inv. semigroup of local bisections

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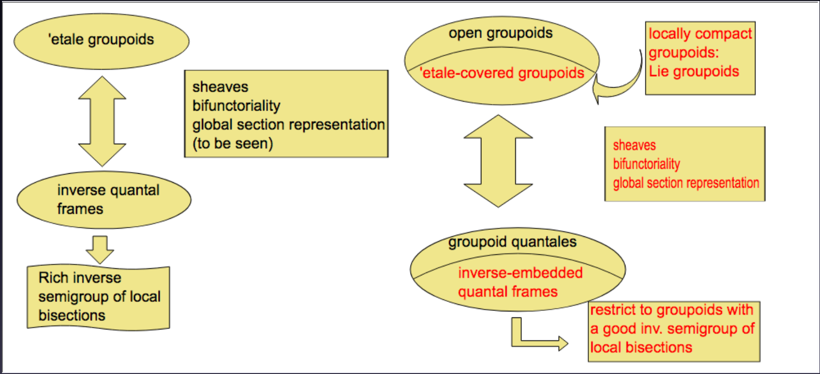
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Preliminaries - Open groupoids and quantales

Definition

An involutive quantale Q is an involutive semigroup,

- $(ab)c = a(bc)$
- $a^{**} = a$
- $(ab)^* = b^* a^*$
- $ae = a \;\&\; ea = a$ (unital quantale)

in the category SL:

- $(\bigvee a_i)b = \bigvee a_i b$
- $b(\bigvee a_i) = \bigvee b a_i$
- $(\bigvee a_i)^* = \bigvee a_i^*$

Preliminaries - Open groupoids and quantales

$$G = \quad G_2 \xrightarrow{\;m\;} G_1 \overset{i}{\circlearrowleft} \; \begin{array}{c} \xrightarrow{\;r\;} \\ \xleftarrow{\;u\;} \\ \xrightarrow[\;d\;]{} \end{array} G_0$$

$$\begin{array}{ccc} G_2 & \xrightarrow{\pi_1} & G_1 \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle r} \\ G_1 & \xrightarrow[d]{} & G_0 \end{array} \qquad \begin{array}{ccc} G_2 & \xrightarrow{\pi_1} & G_1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle d} \\ G_1 & \xrightarrow[d]{} & G_0 \end{array}$$

Lemma

- *d open $\Rightarrow$ m open (open grpd).*
- *d étale $\Rightarrow$ m étale (étale grpd $\Leftrightarrow$ u and d are open) .*

In *SL*:

$$G_1 \otimes G_1 \xrightarrow{\;q\;} G_2 \xrightarrow{\;m_!\;} G_1 \; , \quad G_1 \xrightarrow[\cong]{\;i_!\;} G_1$$

this defines a (unital) involutive quantale $\mathcal{O}(G)$:

- $ab = m_!(q(a \otimes b)), \qquad a^* = i_!(a), \qquad e = u_!(1_{G_0})$

Preliminaries - Open groupoids and quantales

Definition

The involutive quantales of this type are groupoid quantales. In particular, if they are unital then they are inverse quantal frames. In this case we have $\downarrow(e) \cong G_0$. Let us denote this locale by B (the base locale).

Étale-covered groupoids

Definition (Étale-covered groupoids)

By an étale-covered groupoid is meant an open groupoid G equipped with an étale groupoid $\widehat{G}$ such that there is a surjective functor of groupoids $J : \widehat{G}_1 \to G_1$ such that $J_0 : \widehat{G}_0 \to G_0$ is an isomorphism.

Definition (Inverse-embedded quantal frames)

By an *inverse-embedded quantal frame* is meant a (non-unital) involutive quantale frame $\mathcal{O}$ together with an inverse quantal frame $\widehat{\mathcal{O}}$ such that there exists an embedding map of involutive quantal frames $j : \mathcal{O} \to \widehat{\mathcal{O}}$ which is also an $\widehat{\mathcal{O}}$-bimodule homomorphism, satisfying: $\mathcal{O} \otimes_B \mathcal{O} \xrightarrow{j \otimes id} \widehat{\mathcal{O}} \otimes_B \mathcal{O}$ is mono.

Functoriality: G-bundles

$$\begin{array}{c} G_1 \\ d \downarrow \; \downarrow r \\ G_0 \end{array} \quad \text{acts on} \quad \begin{array}{c} X \\ \downarrow p \\ G_0 \end{array}$$

$$\begin{array}{ccc} G_1 \times_{G_0} X & \xrightarrow{\;a\;} & X \\ \pi_1 \downarrow & \text{pullback} & \downarrow p \\ G_1 & \xrightarrow[\;d\;]{} & G_0 \end{array}$$

d open $\Rightarrow$ a open.

Functoriality: G-bundles

X can be regarded as a left $\mathcal{O}(G)$-module (denoted by $\mathcal{O}(X)$):

In SL: $\mathcal{O}(G_1) \otimes \mathcal{O}(X) \xrightarrow{q} \mathcal{O}(G_1) \otimes_B \mathcal{O}(X) \xrightarrow{a_!} \mathcal{O}(X)$

Theorem

An $\mathcal{O}(G)$-module M is of the form $\mathcal{O}(X)$ iff it is an $\mathcal{O}(\widehat{G})$-locale; that is:

- *M is a locale,*
- *$bx = b1 \wedge x$ for all $b \in B$ and $x \in \mathcal{O}(X)$ (bundle condition)*

and in addition it has to satisfy the following condition:

$$a^*(\mathcal{O}(X)) \subset \mathcal{O}(G) \otimes_B \mathcal{O}(X) \quad \text{(descent condition)}$$

$\mathcal{O}(X)$ shall be called an $\mathcal{O}(G)$-locale.

Functoriality: G-bundles

Definition

Let $\mathcal{O}(G)$-*Loc* be category of $\mathcal{O}(G)$-locales whose morphisms are the maps of locales f such that f^* is a homomorphism of $\mathcal{O}(G)$-modules.

Theorem

The $\mathcal{O}(G)$-Loc is isomorphic to the category of G-bundles G-Loc.

Proof.

- $\mathcal{O}(\widehat{G})$-*Loc* $\cong \widehat{G}$-*Loc* (Resende).
- The descent condition ensures that any $\widehat{G}$-bundle is a G-bundle.

□

Functoriality: G-sheaves

The $\mathcal{O}(G)$-locales that correspond to groupoid sheaves (p is a local homeomorphism) have the following characterization:

Theorem

An $\mathcal{O}(G)$-module M corresponds to a G-sheaf if and only if it is an $\mathcal{O}(\widehat{G})$-sheaf; that is:

- *M is a complete Hilbert $\mathcal{O}(\widehat{G})$-module with an inner product $\langle -, - \rangle : M \times M \to \mathcal{O}(\widehat{G})$ and Hilbert basis Γ.*

and in addition it satisfies the following condition:

$$\langle -, - \rangle \in \mathcal{O}(G)$$

Corollary

$$BG \cong \mathcal{O}(G){-}Sh$$

Functoriality: Principal G-bundles

A G-bundle over M:

$$G_0 \xleftarrow{\;p\;} X \xrightarrow{\;\pi\;} M \quad \text{s.t} \quad \begin{array}{ccc} G_1 \times_{G_0} X & \xrightarrow{\;a\;} & X \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\;\pi\;]{} & M. \end{array}$$

and if

$$\langle a, \pi_2 \rangle : G_1 \times_{G_0} X \xrightarrow{\;\cong\;} X \times_M X \quad \& \quad \pi : X \to M \quad \text{open surjection}$$

then it shall be called a principal G-bundle over M:

Lemma

- *$M \cong X/G$ (the orbit locale).*
- *$\mathcal{O}(X/G) = \mathcal{O}(X/\widehat{G}) := I(X)$ (the invariant elements).*
- *G étale $\Rightarrow \pi : X \to M$ local homeomorphism.*

Functoriality: Principal G-bundles

The $\mathcal{O}(G)$-locales that correspond to principal G-bundles over X/G have the following characterization:

Theorem

An $\mathcal{O}(G)$-module X corresponds to a principal G-bundle over X/G if and only if it is a principal $\mathcal{O}(G)$-locale; that is:

- *The groupoid quantale $\mathcal{O}(\widetilde{G})$ of the action groupoid $\widetilde{G} = G_1 \times_{G_0} X \underset{\pi_2}{\overset{a}{\rightrightarrows}} X$ is a principal quantale, that is:*

$$\mathcal{O}(\widetilde{G}) \cong R(\mathcal{O}(\widetilde{G})) \otimes_{T(\mathcal{O}(\widetilde{G}))} L(\mathcal{O}(\widetilde{G}))$$

Functoriality: H-S maps

Definition

Let G and H be groupoids. A bibundle from G to H is a locale X, equipped with a left G-bundle structure and a right H-bundle structure that are compatible in the natural way:

$$
\begin{array}{ccc}
G_1 \times_{G_0} X & \xrightarrow{\;a\;} & X \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle q} \\
X & \xrightarrow[\;q\;]{} & H_0
\end{array}
\qquad
\begin{array}{ccc}
X \times_{H_0} H_1 & \xrightarrow{\;b\;} & X \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p} \\
X & \xrightarrow[\;p\;]{} & G_0
\end{array}
$$

$$
\begin{array}{ccc}
G_1 \times_{G_0} X \times_{H_0} H_1 & \xrightarrow{\;a \times id_{H_1}\;} & X \times_{H_0} H_1 \\
{\scriptstyle id_{G_1} \times b}\downarrow & & \downarrow{\scriptstyle b} \\
G_1 \times_{G_0} X & \xrightarrow[\;a\;]{} & X
\end{array}
$$

Functoriality: H-S maps

Definition

By an $\mathcal{O}(G)$-$\mathcal{O}(H)$-bilocale is meant an $\mathcal{O}(G)$-$\mathcal{O}(H)$-bimodule M that is a locale which satisfies the bundle and the descent conditions wrt both actions.

Theorem

1. *The category of G-H-bibundles is isomorphic to the category of $\mathcal{O}(G)$-$\mathcal{O}(H)$-bilocales.*
2. *The bicategory of étale-covered groupoids is equivalent to the bicategory of inverse-embedded quantal frames.*

Functoriality: H-S maps

A Hilsum–Skandalis map from G to H is (the isomorphism class of) a principal G-H-bibundle X, i.e., a G-H-bibundle X such that the left G-bundle is a principal G-bundle over H_0.

Lemma

A Hilsum–Skandalis map from G to H is the same as a(n isomorphism class of a) principal $\mathcal{O}(G)$-$\mathcal{O}(H)$-bilocale.

Theorem

The previous bi-equivalence restricts to a bi-equivalence between the categories of Hilsum–Skandalis maps for étale-covered groupoids and inverse-embedded quantal frames, respectively.

Functoriality: Morita equivalence

Definition

G and H are said to be Morita equivalent if they are isomorphic in the category HS maps; that is, there is a HS map X from G to H and a HS map Y from H to G such that

$$X \otimes_G Y \cong H \quad \& \quad Y \otimes_H X \cong G\,,$$

as bilocales in *Loc*.

Lemma

G and H are Morita equivalent if and only if $\mathcal{O}(G)$ and $\mathcal{O}(H)$ are isomorphic in H–S maps; that is, there exists a principal $\mathcal{O}(G)$-$\mathcal{O}(H)$-bilocale X and a principal $\mathcal{O}(H)$-$\mathcal{O}(G)$-bilocale Y such that:

$$X \otimes_{\mathcal{O}(\widehat{G})} Y \cong \mathcal{O}(H) \quad \& \quad Y \otimes_{\mathcal{O}(\widehat{H})} X \cong \mathcal{O}(G)\,, \quad \textit{in Frm.}$$

Functoriality: Morita equivalence

Definition

By a principal $\mathcal{O}$-sheaf is meant to be an $\mathcal{O}$-locale X which is both a principal $\mathcal{O}$-locale and an $\mathcal{O}$-sheaf.

Theorem

X is a principal $\mathcal{O}$-sheaf if and only if X is an $\widehat{\mathcal{O}}$-sheaf satisfying the following condition:

$$\forall_{q\in\mathcal{O}}\ \forall_{(s,t)\in\Gamma_X\times_q\Gamma_X}\ \langle s,t\rangle = q\ ,$$

where $\Gamma_X \times_q \Gamma_X = \left\{(s,t)\in\Gamma_X^2 \mid p_!(s) = d_!(q)\, , p_!(t) = r_!(q)\, ,\ \exists_{u\in\mathcal{I}(\widehat{\mathcal{O}})\cap\downarrow(q)}\ s = ut\right\}$.

Corollary

X is a principal G-sheaf if and only if X is a principal $\mathcal{O}(G)$-sheaf.

Functoriality: Morita equivalence

The inverse of an isomorphism X in the category of HS maps can always be taken to be X^* (the dual of X):

Theorem

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be inverse-embedded quantal frames and X be a biprincipal $\mathcal{O}_1$-$\mathcal{O}_2$-sheaf. Then (⇔ étale case)

$$X \otimes_{\widehat{\mathcal{O}}_2} X^* \cong \mathcal{O}_1 \quad \& \quad X^* \otimes_{\widehat{\mathcal{O}}_1} X \cong \mathcal{O}_2\,, \quad \textit{in Frm.}$$

Therefore $\mathcal{O}_1$ and $\mathcal{O}_2$ are Morita equivalent.

What does G look like in BG?

Theorem

Let G be an étale groupoid, and let $\mathbf{G}$ be the domain map $d : G_1 \to G_0$ equipped with the left G-action given by multiplication, regarded as an object of BG. Then the powerobject $P(\mathbf{G} \times \mathbf{G})$ is an internal quantale in BG (the internal quantale of binary relations on $\mathbf{G}$), and we have

$$\hom_{BG}(1, P(\mathbf{G} \times \mathbf{G})) \cong \mathcal{O}(G)\ .$$

A generalization of this for general open groupoids is unlikely to exist, but for an étale-covered groupoid G we have the following theorem (from which the previous one arises as a corollary):

What does G look like in BG?

Theorem

Let G be an étale-covered groupoid, and let $\mathbf{G}$ be the domain map $d : G_1 \to G_0$ equipped with its left G-action, regarded as an internal locale in BG. Then $\mathbf{G} \otimes \mathbf{G}$ is an internal involutive quantale in BG, and there is an isomorphism of involutive quantales

$$\hom_{BG}(1, \mathbf{G} \otimes \mathbf{G}) \cong \mathcal{O}(G) .$$

- from the pair $(BG, \mathbf{G})$ one can reconstruct the étale-covered groupoid G up to isomorphisms.
- this construction of G is certainly not equivalent to the construction of groupoids from toposes via descent, since the latter always yields étale-complete groupoids, and thus, in particular, it excludes simply connected Lie groups.

Thanks for your attention.