Stability properties for n-permutable categories

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  (iv) $R^n \leqslant R^{n-1}$, $\forall$ reflexive relation R (transitivity)

Embedding Theorem I

- Thm [HM] $\mathbb{V}$ is an n-permutable variety iff $\exists$ ternary operations

$$\begin{cases} w_1(x,y,y) = x \\ w_i(x,x,y) = w_{i+1}(x,y,y), \quad 2 \leqslant i \leqslant n-2 \\ w_{n-1}(x,x,y) = y \\ \end{cases}$$

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- a partial operation $\pi^s \colon \bar{s} \to s$ defined on the image of α^s

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- Thm [HM] $\mathbb{V}$ is an n-permutable variety iff $\exists$ $(n+1)$-ary operations

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γ, β, δ regular epis $\Rightarrow$ λ regular epi

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X \arrow[uu, shift right] \arrow[rr, dashed, two heads, "\gamma"] \arrow[dr, shift right, "f"'] & & U \arrow[uu, dashed, shift right] \arrow[dr, dashed, shift right, "h"'] & \\
& Y \arrow[ul, shift right] \arrow[uu, shift right] \arrow[rr, two heads, "\beta"'] & & V \arrow[ul, dashed, shift right] \arrow[uu, shift right]
\end{tikzcd}$$

γ, β, δ regular epis $\Rightarrow$ λ regular epi

- 3-permutable varieties:

Stability property $n = 3$

- Prop [GR] $\mathbb{C}$ is a (regular) Goursat cat iff for any

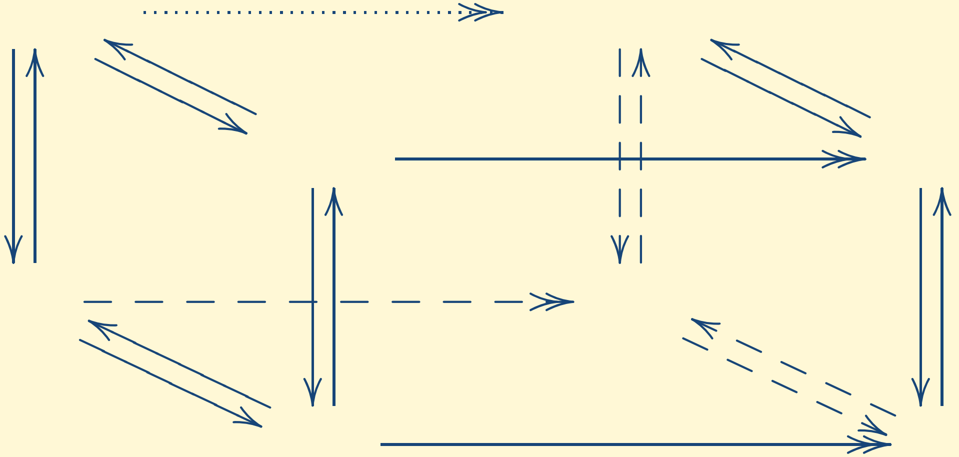
$$\begin{tikzcd}
X \times_Y Z \arrow[rr, dotted, two heads, "\lambda"] \arrow[dr, shift left] \arrow[dd, shift right] & & U \times_V W \arrow[dr, shift left] \arrow[dd, dashed, shift right] & \\
& Z \arrow[ul, shift left] \arrow[rr, two heads, "\delta"] \arrow[dd, shift right, "g"'] & & W \arrow[ul, shift left] \arrow[dd, shift right, "k"'] \\
X \arrow[uu, shift right] \arrow[rr, dashed, two heads, "\gamma"] \arrow[dr, shift left, "f"'] & & U \arrow[uu, dashed, shift right] \arrow[dr, dashed, shift left, "h"'] & \\
& Y \arrow[uu, shift right] \arrow[ul, shift left] \arrow[rr, two heads, "\beta"'] & & V \arrow[uu, shift right] \arrow[ul, dashed, shift left]
\end{tikzcd}$$

γ, β, δ regular epis $\Rightarrow$ λ regular epi

- 3-permutable varieties: F - free algebra on one element

$$\nabla = (1_F \; 1_F) \colon 2F \to F$$

Stability property $n = 3$



- $\mathbf{3}$-permutable varieties: $\boldsymbol{F}$ - free algebra on one element

$$\boldsymbol{\nabla} = (\mathbf{1}_{\boldsymbol{F}}\ \mathbf{1}_{\boldsymbol{F}})\colon \mathbf{2}\boldsymbol{F} \to \boldsymbol{F}$$

Stability property $n = 3$

$$\begin{tikzcd}
P \arrow[rr, dotted, two heads] \arrow[dr, shift right] \arrow[dd, shift right] & & \mathrm{Eq}(\nabla) \arrow[dr, shift right] \arrow[dd, dashed, shift right] & \\
& 3F \arrow[ul, shift right] \arrow[rr, two heads, "\nabla + 1_F"] \arrow[dd, shift right, "1_F + \nabla"'] & & 2F \arrow[ul, shift right] \arrow[dd, shift right, "\nabla"'] \\
3F \arrow[uu, shift right] \arrow[rr, dashed, two heads, "1_F + \nabla"'] \arrow[dr, shift right, "\nabla + 1_F"'] & & 2F \arrow[uu, dashed, shift right] \arrow[dr, dashed, two heads, "\nabla"'] & \\
& 2F \arrow[ul, shift right] \arrow[uu, shift right] \arrow[rr, two heads, "\nabla"'] & & F \arrow[uu, shift right] \arrow[ul, dashed, shift right]
\end{tikzcd}$$

- $\mathbf{3}$-permutable varieties: F - free algebra on one element

$$\nabla = (1_F\ 1_F) \colon 2F \to F$$

Stability property $n = 3$

$$\begin{tikzcd}
P \arrow[rr, dotted, two heads, "\text{surjective}"] \arrow[dr, shift right] \arrow[dd, shift right] & & \mathrm{Eq}(\nabla) \arrow[dr, shift right] \arrow[dd, dashed, shift right] & \\
& 3F \arrow[ul, shift right] \arrow[rr, two heads, "\nabla + 1_F"] \arrow[dd, shift right, "1_F + \nabla"'] & & 2F \arrow[ul, shift right] \arrow[dd, shift right, "\nabla"'] \\
3F \arrow[uu, shift right] \arrow[rr, dashed, two heads, "1_F + \nabla"'] \arrow[dr, shift right, "\nabla + 1_F"'] & & 2F \arrow[uu, dashed, shift right] \arrow[dr, dashed, shift right, "\nabla"'] & \\
& 2F \arrow[uu, shift right] \arrow[ul, shift right] \arrow[rr, two heads, "\nabla"'] & & F \arrow[uu, shift right] \arrow[ul, dashed, shift right]
\end{tikzcd}$$

- $\mathbf{3}$-permutable varieties: $\boldsymbol{F}$ - free algebra on one element

$$\nabla = (1_F\ 1_F)\colon 2F \to F$$

Stability property $n = 3$

$$\begin{cases} p_1(x,y) = x \\ p_2(x,y) = y \end{cases}$$

$$\begin{tikzcd}
 & & & (p_1,p_2) \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \\
P \arrow[rrr, dotted, two heads, "\text{surjective}"] \arrow[dd, shift right] \arrow[dr, shift right] & & & \mathrm{Eq}(\nabla) \arrow[dd, dashed, shift right] \arrow[dr, shift right] & \\
 & 3F \arrow[ul, shift right] \arrow[rrr, two heads, "\nabla + 1_F"] \arrow[dd, shift right, "1_F + \nabla"'] & & & 2F \arrow[ul, shift right] \arrow[dd, shift right, "\nabla"'] \\
3F \arrow[uu, shift right] \arrow[rrr, dashed, two heads, "1_F + \nabla"'] \arrow[dr, shift right, "\nabla + 1_F"'] & & & 2F \arrow[uu, dashed, shift right] \arrow[dr, dashed, shift right, "\nabla"'] & \\
 & 2F \arrow[ul, shift right] \arrow[uu, shift right] \arrow[rrr, two heads, "\nabla"'] & & & F \arrow[uu, shift right] \arrow[ul, dashed, shift right]
\end{tikzcd}$$

- 3-permutable varieties: F - free algebra on one element

$$\nabla = (1_F\ 1_F) \colon 2F \to F$$

Stability property $n = 3$

$$\begin{tikzcd}
(r,s) \arrow[d, phantom, "\pitchfork" description] & & (p_1,p_2) \arrow[d, phantom, "\pitchfork" description] & \\
P \arrow[rr, dotted, two heads, "\text{surjective}"] \arrow[dr, shift left] \arrow[dd, shift right] & & \mathrm{Eq}(\nabla) \arrow[dr, shift left] \arrow[dd, dashed, shift right] & \\
& 3F \arrow[ul, shift left] \arrow[rr, two heads, "\nabla + 1_F"] \arrow[dd, shift right, "1_F + \nabla"'] & & 2F \arrow[ul, shift left] \arrow[dd, shift right, "\nabla"'] \\
3F \arrow[uu, shift right] \arrow[rr, dashed, two heads, "1_F + \nabla"'] \arrow[dr, shift left, "\nabla + 1_F"'] & & 2F \arrow[uu, dashed, shift right] \arrow[dr, dashed, shift left, "\nabla"'] & \\
& 2F \arrow[ul, shift left] \arrow[uu, shift right] \arrow[rr, two heads, "\nabla"'] & & F \arrow[ul, dashed, shift left] \arrow[uu, shift right]
\end{tikzcd}$$

$$\begin{cases} p_1(x,y) = x \\ p_2(x,y) = y \end{cases}$$

- 3-permutable varieties: F - free algebra on one element

$$\nabla = (1_F \ 1_F) \colon 2F \to F$$

Stability property $n = 3$

$$\begin{cases} p_1(x,y) = x \\ p_2(x,y) = y \end{cases}$$

$$\begin{tikzcd}
(r,s) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & & (p_1,p_2) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \\
P \arrow[rrr, dotted, two heads, "\text{surjective}"] \arrow[dr, shift left] \arrow[dd, shift right] & & & \mathrm{Eq}(\nabla) \arrow[dr, shift left] \arrow[dd, dashed, shift right] & \\
& 3F \arrow[ul, shift left] \arrow[rrr, two heads, "\nabla + 1_F"] \arrow[dd, shift right, "1_F + \nabla"'] & & & 2F \arrow[ul, shift left] \arrow[dd, shift right, "\nabla"'] \\
3F \arrow[uu, shift right] \arrow[rrr, dashed, two heads, "1_F + \nabla"'] \arrow[dr, shift right, "\nabla + 1_F"'] & & & 2F \arrow[uu, dashed, shift right] \arrow[dr, dashed, two heads, shift left, "\nabla"'] & \\
& 2F \arrow[uu, shift right] \arrow[ul, shift right] \arrow[rrr, two heads, "\nabla"'] & & & F \arrow[uu, shift right] \arrow[ul, dashed, shift left]
\end{tikzcd}$$

$$r(x,x,y) = s(x,y,y)$$

- 3-permutable varieties: F - free algebra on one element

$$\nabla = (1_F \; 1_F) \colon 2F \to F$$

Stability property $n = 3$

$$\begin{tikzcd}
(r,s) & & (p_1,p_2) & \\
\rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} & \\
P \arrow[rr, dotted, two heads, "\text{surjective}"] \arrow[dr, shift left] \arrow[dd, shift right] & & \mathrm{Eq}(\nabla) \arrow[dr, shift left] \arrow[dd, dashed, shift right] & \\
& 3F \arrow[ul, shift left] \arrow[rr, two heads, "\nabla + 1_F"] \arrow[dd, shift right, "1_F + \nabla"'] & & 2F \arrow[ul, shift left] \arrow[dd, shift right, "\nabla"'] \\
3F \arrow[uu, shift right] \arrow[rr, dashed, two heads, "1_F + \nabla"'] \arrow[dr, shift right, "\nabla + 1_F"'] & & 2F \arrow[uu, dashed, shift right] \arrow[dr, dashed, shift left, "\nabla"'] & \\
& 2F \arrow[uu, shift right] \arrow[ul, shift right] \arrow[rr, two heads, "\nabla"'] & & F \arrow[uu, shift right] \arrow[ul, dashed, shift left]
\end{tikzcd}$$

$$\begin{cases} p_1(x,y) = x \\ p_2(x,y) = y \end{cases}$$

$$r(x,x,y) = s(x,y,y), \;\; r(x,y,y) = x \;\; \text{and} \;\; s(x,x,y) = y$$

- 3-permutable varieties: F - free algebra on one element

$$\nabla = (1_F \; 1_F) \colon 2F \to F$$

Stability property $n \geqslant 3$

- **Thm** $\mathbb{C}$ regular + binary sums. $\mathbb{C}$ is an n-permutable cat iff for any

$$\begin{tikzcd}
& & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"'] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\lambda"] & & & & & U \times_V W \arrow[dl, dotted, shift left] \arrow[ddddddrr, shift left] & & \\
X_1 \arrow[rrrrrrrrr, dashed, two heads, "\gamma"] \arrow[ddr, shift right, "f_1"'] & & & & & & & & & U \arrow[ur, dotted, shift left] \arrow[ddddddrr, dashed, shift right, "h"'] & & & \\
& & X_2 \arrow[ddr, shift right, "f_2"'] \arrow[dl, shift right, "g_1"'] & & & & & & & & & & \\
& Y_1 \arrow[uul, shift right, "s_1"'] \arrow[ur, shift right, "t_1"'] \arrow[ddddrrrrrrrrrr, dashed, "\beta_1"] & & & X_3 \arrow[dl, shift right, "g_2"'] & & & & & & & & \\
& & & Y_2 \arrow[uul, shift right, "s_2"'] \arrow[ur, shift right, "t_2"'] \arrow[dddrrrrrrrr, dashed, "\beta_2"] & \ddots & & & & & & & & \\
& & & & \ddots & & X_{n-2} \arrow[ddr, shift right, "f_{n-2}"'] & & & & & & \\
& & & & & & & & X_{n-1} \arrow[dl, shift right, "g_{n-2}"'] \arrow[rrrr, two heads, "\delta"] & & & & W \arrow[uuuuuull, shift left] \arrow[dl, shift right, "k"'] \\
& & & & & & & Y_{n-2} \arrow[uul, shift right, "s_{n-2}"'] \arrow[ur, shift right, "t_{n-2}"'] \arrow[rrrr, two heads, "\beta_{n-2}"'] & & & \cdots & V \arrow[uuuuuull, dashed, shift right, "u"'] \arrow[ur, shift right, "w"'] &
\end{tikzcd}$$

Stability property $n \geqslant 3$

- **Thm** $\mathbb{C}$ regular + binary sums. $\mathbb{C}$ is an n-permutable cat iff for any

$$\begin{tikzcd}
 & & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"'] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\lambda"] & & & & & U \times_V W \arrow[dl, dotted, shift right] \arrow[ddddddr, shift left] & \\
X_1 \arrow[rrrrrrrrr, dashed, two heads, "\gamma"] \arrow[ddr, shift right, "f_1"'] & & & & & & & & & U \arrow[ur, dotted, shift right] \arrow[ddddddr, dashed, shift right, "h"'] & & \\
 & & X_2 \arrow[dl, shift right, "g_1"'] \arrow[ddr, shift right, "f_2"'] & & & & & & & & & \\
 & Y_1 \arrow[uul, shift right, "s_1"'] \arrow[ur, shift right, "t_1"'] \arrow[ddddrrrrrrrrr, dashed, "\beta_1"] & & & X_3 \arrow[dl, shift right, "g_2"'] & & & & & & & \\
 & & & Y_2 \arrow[uul, shift right, "s_2"'] \arrow[ur, shift right, "t_2"'] \arrow[dddrrrrrrr, dashed, "\beta_2"] & & \ddots & & & & & & \\
 & & & & & \ddots & X_{n-2} \arrow[ddr, shift right, "f_{n-2}"'] & & & & & \\
 & & & & & & & & X_{n-1} \arrow[dl, shift right, "g_{n-2}"'] \arrow[rrr, two heads, "\delta"] & & & W \arrow[uuuuuul, shift left] \arrow[dl, shift right, "k"'] \\
 & & & & & & & Y_{n-2} \arrow[uul, shift right, "s_{n-2}"'] \arrow[ur, shift right, "t_{n-2}"'] \arrow[rrr, "\beta_{n-2}"'] & & \cdots & V \arrow[uuuuuul, dashed, shift right, "u"'] \arrow[ur, shift right, "w"'] &
\end{tikzcd}$$

(β's split epis + commutativity conds)

Stability property $n \geqslant 3$

- Thm $\mathbb{C}$ regular + binary sums. $\mathbb{C}$ is an n-permutable cat iff for any

$$\begin{tikzcd}
 & & & & & L \arrow[rrrrr, dotted, two heads, "\lambda"] \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrrr, "\pi_{n-1}"] & & & & & U \times_V W \arrow[dl, dotted, shift left] \arrow[ddddddrr, shift left] & & \\
X_1 \arrow[rrrrrrrrr, dashed, two heads, "\gamma"] \arrow[ddr, shift right, "f_1"'] & & & & & & & & & U \arrow[ur, dotted, shift left] \arrow[ddddddrr, dashed, shift right, "h"'] & & & \\
 & & X_2 \arrow[dl, shift right, "g_1"'] \arrow[ddr, shift right, "f_2"'] & & & & & & & & & & \\
 & Y_1 \arrow[uul, shift right, "s_1"'] \arrow[ur, shift right, "t_1"'] \arrow[ddddrrrrrrrrrr, dashed, "\beta_1"] & & & X_3 \arrow[dl, shift right, "g_2"'] & & & & & & & & \\
 & & & Y_2 \arrow[uul, shift right, "s_2"'] \arrow[ur, shift right, "t_2"'] \arrow[dddrrrrrrrr, dashed, "\beta_2"] & \ddots & & & & & & & & \\
 & & & & \ddots & & X_{n-2} \arrow[ddr, shift right, "f_{n-2}"'] & & & & & & \\
 & & & & & & & & X_{n-1} \arrow[dl, shift right, "g_{n-2}"'] \arrow[rrrr, two heads, "\delta"] & & & & W \arrow[dl, shift right, "k"'] \arrow[uuuuuull, shift left] \\
 & & & & & & & Y_{n-2} \arrow[uul, shift right, "s_{n-2}"'] \arrow[ur, shift right, "t_{n-2}"'] \arrow[rrrr, "\beta_{n-2}"'] & & & \cdots & V \arrow[ur, shift right, "w"'] \arrow[uuuuuull, dashed, shift right, "u"'] &
\end{tikzcd}$$

(β's split epis + commutativity conds)

γ, δ regular epis $\Rightarrow$ λ regular epi

Stability property $n \geqslant 3$

- **Thm** $\mathbb{C}$ regular + binary sums. $\mathbb{C}$ is an n-permutable cat iff for any

$$\begin{tikzcd}
 & & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"'] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[dddddddrrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\lambda"] & & & & & U \times_V W \arrow[dl, dotted] \arrow[dddddddr, shift left] \\
X \arrow[rrrrrrrrr, dashed, two heads, "\gamma"] \arrow[ddr, shift right, "f"'] & & & & & & & & & U \arrow[ur, dotted] \arrow[dddddddr, dashed, shift right, "h"'] \arrow[dddddddr, dashed, shift left, "u"] & \\
 & & X \arrow[dl, shift right, "g"'] \arrow[ddr, shift right, "f"'] & & & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddddrrrrrrrrr, dashed, "\beta"] & & & X \arrow[dl, shift right, "g"'] & & & & & & & \\
 & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddrrrrrrr, dashed, "\cdots\ \beta"'] & & \ddots & & & & & & \\
 & & & & & & X \arrow[dddr, shift right, "f"'] & & & & & \\
 & & & & & \ddots & & & & & & \\
 & & & & & & & & X \arrow[dl, shift right, "g"'] \arrow[rrr, "\delta"] & & & W \arrow[uuuuuuul, shift left] \arrow[dl, shift right, "k"'] \\
 & & & & & & & Y \arrow[uuul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrr, two heads, "\beta"'] & & & V \arrow[ur, shift right, "w"'] &
\end{tikzcd}$$

γ, δ regular epis $\Rightarrow$ $\boldsymbol{\lambda}$ regular epi

Stability property $n \geqslant 3$

- **Thm** $\mathbb{C}$ regular + binary sums. $\mathbb{C}$ is an n-permutable cat iff for any

$$\begin{tikzcd}
 & & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"'] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"'] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrrrrr, dotted, two heads, "\lambda"] & & & & & & U \times_V W \arrow[dl, dotted] \arrow[ddddddrr, shift left] & & \\
X \arrow[ddr, shift right, "f"'] \arrow[rrrrrrrrrr, dashed, two heads, "\gamma"] & & & & & & & & & & U \arrow[ur, dotted] \arrow[ddddddrr, dashed, shift right, "h"'] & & & \\
 & & X \arrow[dl, shift right, "g"'] \arrow[ddr, shift right, "f"'] & & & & & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddrrrrrrrrrrr, dashed, "\beta"] & & & X \arrow[dl, shift right, "g"'] & & & & & & & & & \\
 & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddrrrrrrrrr, dashed, "\beta"] & & \ddots & & & & & & & & \\
 & & & & & \ddots & X \arrow[ddr, shift right, "f"'] & & & & & & & \\
 & & & & & & & & X \arrow[dl, shift right, "g"'] \arrow[rrrrr, "\delta"] & & & & & W \arrow[dl, shift right, "k"'] \arrow[uuuuuull, shift left] \\
 & & & & & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrrrr, "\beta"'] & & & & \cdots & V \arrow[uuuuuull, dashed, shift right, "u"'] \arrow[ur, shift right, "w"'] &
\end{tikzcd}$$

$\Rightarrow$ **Embedding Theorem** (follows varietal proof)

Stability property $n \geqslant 3$

- **Thm** $\mathbb{C}$ regular + binary sums. $\mathbb{C}$ is an n-permutable cat iff for any

$$\begin{tikzcd}
 & & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"'] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\lambda"] & & & & & U \times_V W \arrow[dl, dotted, shift left] \arrow[ddddddrr, shift left] & & \\
X \arrow[rrrrrrrrr, dashed, "\gamma"] \arrow[ddr, shift right, "f"'] & & & & & & & & & U \arrow[ur, dotted, shift left] \arrow[ddddddrr, dashed, shift right, "h"'] & & & \\
 & & X \arrow[dl, shift right, "g"'] \arrow[ddr, shift right, "f"'] & & & & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddrrrrrrrrrr, dashed, "\beta"] & & & X \arrow[dl, shift right, "g"'] & & & & & & & & \\
 & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddrrrrrrrr, dashed, "\beta"] & \ddots & & & & & & & & \\
 & & & & \ddots & & X \arrow[ddr, shift right, "f"'] & & & & & & \\
 & & & & & & & & X \arrow[dl, shift right, "g"'] \arrow[rrrr, "\delta"] & & & & W \arrow[dl, shift right, "k"'] \arrow[uuuuuull, shift left] \\
 & & & & & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrrr, "\beta"] & & & \cdots \beta & V \arrow[ur, shift right, "w"'] \arrow[uuuuuull, dashed, shift right, "u"'] &
\end{tikzcd}$$

$\Rightarrow$ **Embedding Theorem** (follows varietal proof) - **doesn't use sums**

Stability property $n \geqslant 3$

- Thm $\mathbb{C}$ regular + binary sums. $\mathbb{C}$ is an n-permutable cat iff for any

$$\begin{tikzcd}
 & & & & & L \arrow[lllll, "\pi_1"'] \arrow[dd, lll, "\pi_2"'] \arrow[ddd, l, "\pi_3"] \arrow[ddddd, r, "\pi_{n-2}"] \arrow[ddddddd, rrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\lambda"] & & & & & U \times_V W \arrow[dl, dotted] \arrow[ddddddd, r, shift left] & \\
X \arrow[rrrrrrrrr, dashed, "\gamma"] \arrow[dd, r, shift right, "f"'] & & & & & & & & & U \arrow[ur, dotted] \arrow[ddddddd, r, dashed, shift right, "h"'] & & \\
 & & X \arrow[dl, shift right, "g"'] \arrow[dd, r, shift right, "f"'] & & & & & & & & & \\
 & Y \arrow[uu, l, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddd, rrrrrrrrr, dashed, "\beta"] & & & X \arrow[dl, shift right, "g"'] & & & & & & & \\
 & & & Y \arrow[uu, l, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddd, rrrrrrr, dashed, "\beta"] & & \ddots & & & & & & \\
 & & & & & & X \arrow[ddd, r, shift right, "f"'] & & & & & \\
 & & & & & \ddots & & & & & & \\
 & & & & & & & & X \arrow[dl, shift right, "g"'] \arrow[rrr, "\delta"] & \cdots & & W \arrow[uuuuuuu, l, shift left] \arrow[dl, shift right, "k"'] \\
 & & & & & & & Y \arrow[uuu, l, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrr, "\beta"] & & & V \arrow[uuuuuuu, l, dashed, shift right, "u"'] \arrow[ur, shift right, "w"'] &
\end{tikzcd}$$

$\Rightarrow$ Embedding Theorem (follows varietal proof) - doesn't use sums

$\Leftarrow$ Uses approximate Hagemann-Mitschke co-operations [RVdL]

Stability property $n \geqslant 3$

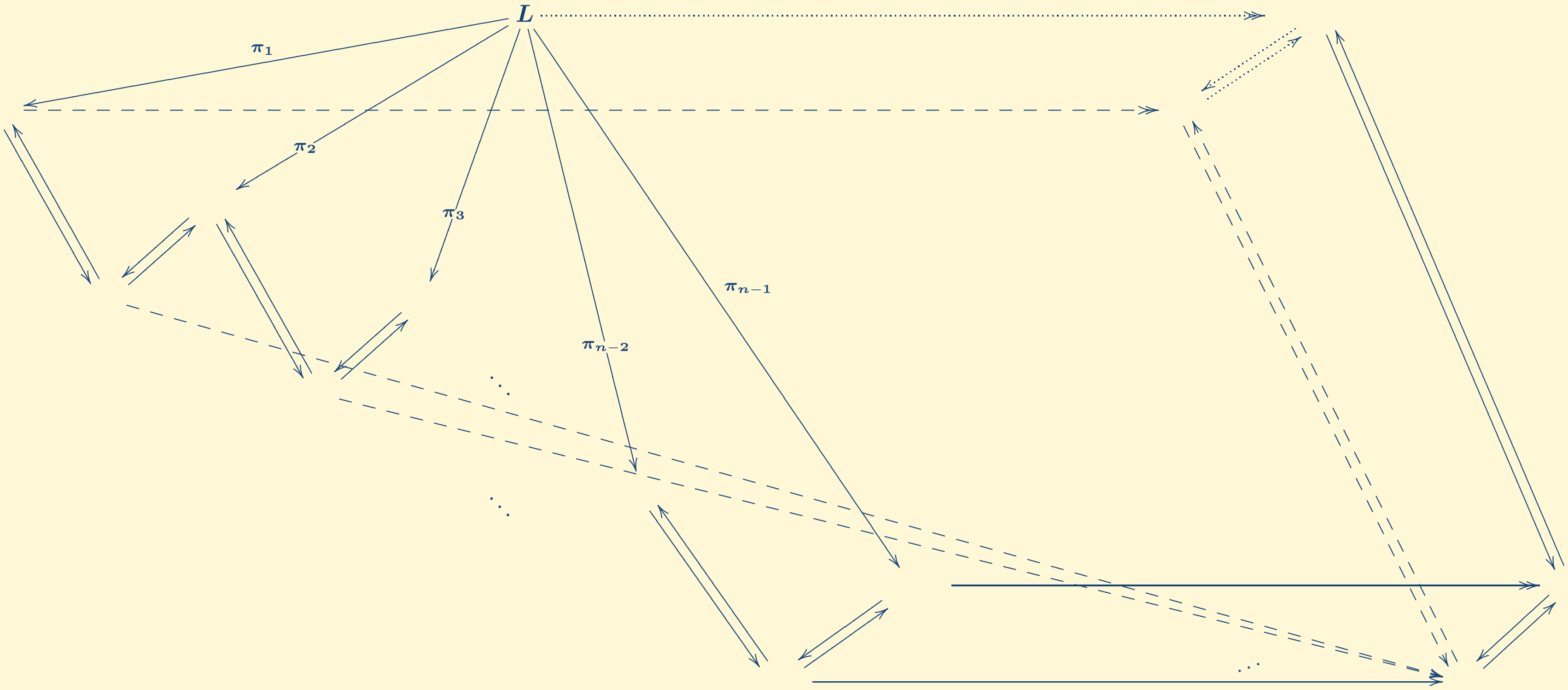
- **Thm** $\mathbb{C}$ regular + binary ? sums. $\mathbb{C}$ is an n-permutable cat iff for any

$$\begin{tikzcd}
 & & & L \arrow[dlll, "\pi_1"'] \arrow[ddll, "\pi_2"] \arrow[dddl, "\pi_3"] \arrow[ddddd, "\pi_{n-2}"] \arrow[ddddddr, "\pi_{n-1}"] \arrow[rrrr, dotted, two heads, "\lambda"] & & & & U \times_V W \arrow[dl, dotted] \arrow[dddddd] \\
X \arrow[dd, shift right, "f"'] \arrow[rrrrrr, dashed, two heads, "\gamma"] & & & & & & U \arrow[ur, dotted] \arrow[ddddddr, dashed, shift right, "h"'] & \\
 & X \arrow[dl, shift right, "g"'] \arrow[dd, shift right, "f"'] & & & & & & \\
Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddrrrrrrr, dashed, "\beta"] & & X \arrow[dl, shift right, "g"'] & & & & & \\
 & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddrrrrrr, dashed, "\beta"] & \ddots & & & & & \\
 & & \ddots & X \arrow[dd, shift right, "f"'] & & & & \\
 & & & & X \arrow[dl, shift right, "g"'] \arrow[rrr, "\delta"] & & & W \arrow[d, shift right, "k"'] \arrow[uuuuuu] \\
 & & & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrrr, "\beta"'] & & & \cdots & V \arrow[u, shift right, "w"'] \arrow[uuuuuul, dashed, shift right, "u"']
\end{tikzcd}$$

$\Rightarrow$ Embedding Theorem (follows varietal proof) - doesn't use sums

$\Leftarrow$ Uses approximate Hagemann-Mitschke co-operations [RVdL]

Stability property $n \geqslant 3$



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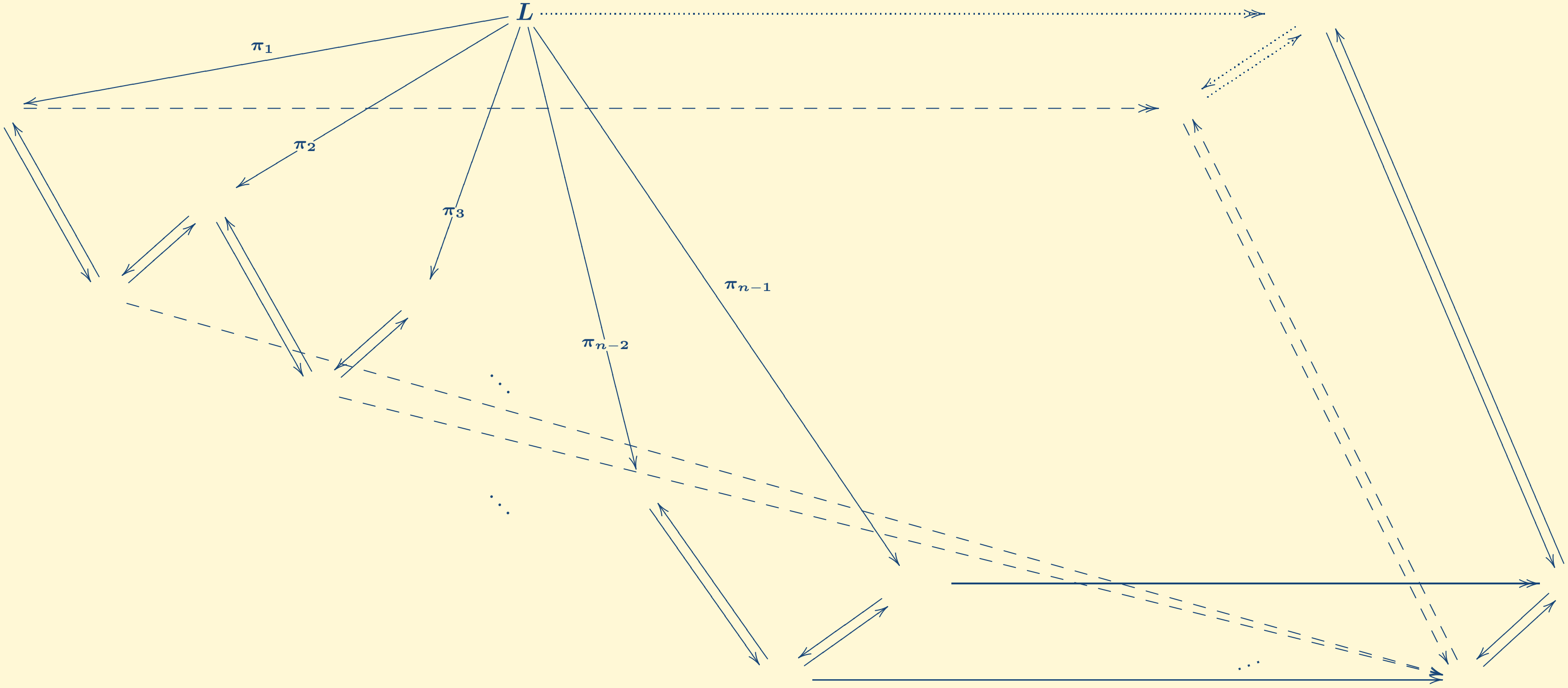

- n-permutable varieties: F - free algebra on one element

$$\nabla = (1_F \; 1_F) \colon 2F \to F$$

Stability property $n \geqslant 3$

$$\begin{tikzcd}
 & & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads] & & & & & \mathbf{Eq}(\nabla) \arrow[dl, dotted, shift right] \arrow[ddddddrr, shift left] & & \\
3F \arrow[rrrrrrrrr, dashed, two heads, "1_F + \nabla"] \arrow[ddr, shift right, "\nabla + 1"'] & & & & & & & & & 2F \arrow[ur, dotted, shift right] \arrow[ddddddrr, dashed, shift left, "\nabla"] & & & \\
 & & 3F \arrow[dl, shift right, "1 + \nabla"'] \arrow[ddr, shift left] & & & & & & & & & & \\
 & 2F \arrow[uul, shift right] \arrow[ur, shift right] \arrow[ddddrrrrrrrrrr, dashed] & & & 3F \arrow[dl, shift right] & & & & & & & & \\
 & & & 2F \arrow[uul, shift left] \arrow[ur, shift right] \arrow[dddrrrrrrrr, dashed, "\nabla"] & & \ddots & & & & & & & \\
 & & & & & \ddots & 3F \arrow[ddr, shift left, "\nabla + 1"'] & & & & & & \\
 & & & & & & & & 3F \arrow[dl, shift right, "1 + \nabla"'] \arrow[rrrr, two heads, "\nabla + 1_F"] & & & & 2F \arrow[uuuuuull, shift left] \arrow[dl, shift right, "\nabla"'] \\
 & & & & & & & 2F \arrow[uul, shift left] \arrow[ur, shift right] \arrow[rrrr, "\nabla"'] & & & \cdots & F \arrow[uuuuuull, dashed, shift left] \arrow[ur, shift right] &
\end{tikzcd}$$

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 & & & & & L \arrow[lllll, "\pi_1"'] \arrow[ddlll, "\pi_2"'] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\text{surjective}"] & & & & & \mathbf{Eq}(\nabla) \arrow[dl, dotted, shift left] \arrow[ddddddrr, shift left] & & \\
3F \arrow[rrrrrrrrr, dashed, two heads, "1_F + \nabla"] \arrow[ddr, shift left, "\nabla + 1"'] & & & & & & & & & 2F \arrow[ur, dotted, shift left] \arrow[ddddddrr, dashed, shift left, "\nabla"] & & & \\
 & & 3F \arrow[dl, shift left] \arrow[ddr, shift left] & & & & & & & & & & \\
 & 2F \arrow[uul, shift left] \arrow[ur, shift left, "1+\nabla"] \arrow[ddddrrrrrrrrrr, dashed] & & & 3F \arrow[dl, shift left] & & & & & & & & \\
 & & & 2F \arrow[uul, shift left] \arrow[ur, shift left] \arrow[dddrrrrrrrr, dashed, "\nabla"] & \ddots & & & & & & & & \\
 & & & & \ddots & & 3F \arrow[ddr, shift left, "\nabla + 1"'] & & & & & & \\
 & & & & & & & & 3F \arrow[dl, shift left] \arrow[rrrr, "\nabla + 1_F"] & & & & 2F \arrow[uuuuuull, shift left] \arrow[dl, shift left] \\
 & & & & & & & 2F \arrow[uul, shift left] \arrow[ur, shift left, "1+\nabla"] \arrow[rrrr, "\nabla"'] & & & \cdots & F \arrow[uuuuuull, dashed, shift left] \arrow[ur, shift left, "\nabla"] &
\end{tikzcd}$$

- n-permutable varieties: F - free algebra on one element

$$\nabla = (1_F\ 1_F) \colon 2F \to F$$

Stability property $n \geqslant 3$

$$\begin{tikzcd}
 & & & & & & & & & & (p_1, p_2) & & \\
 & & & & & & & & & & \rotatebox[origin=c]{90}{$\in$} & & \\
 & & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"'] \arrow[dddl, "\pi_3"'] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\text{surjective}"] & & & & & \mathbf{Eq}(\nabla) \arrow[dl, dotted, shift right] \arrow[ddddddrr, shift left] & & \\
3F \arrow[rrrrrrrrr, dashed, "1_F + \nabla"] \arrow[ddr, shift right, "\nabla + 1"'] & & & & & & & & & 2F \arrow[ur, dotted, shift right] \arrow[ddddddrr, dashed, shift left, "\nabla"] & & & \\
 & & 3F \arrow[dl, shift right] \arrow[ddr, shift left] & & & & & & & & & & \\
 & 2F \arrow[uul, shift right] \arrow[ur, shift right, "1 + \nabla"] \arrow[ddddrrrrrrrrrr, dashed, "\nabla"] & & & 3F \arrow[ddl, shift right] & & & & & & & & \\
 & & & 2F \arrow[uul, shift left] \arrow[ur, shift right] \arrow[dddrrrrrrrr, dashed] & & \ddots & & & & & & & \\
 & & & & & \ddots & 3F \arrow[ddr, shift right, "\nabla + 1"'] & & & & & & \\
 & & & & & & & & 3F \arrow[dl, shift right] \arrow[rrrr, "\nabla + 1_F"] & & & & 2F \arrow[dl, shift right, "\nabla"'] \arrow[uuuuuull, shift left] \\
 & & & & & & & 2F \arrow[uul, shift right] \arrow[ur, shift right, "1 + \nabla"] \arrow[rrrr, "\nabla"'] & & & \cdots & F \arrow[ur, shift right] \arrow[uuuuuull, dashed, shift left] &
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 & & & (w_1, \cdots, w_{n-1}) & & & & (p_1, p_2) & \\
 & & & \Cap & & & & \Cap & \\
 & & & L \arrow[dlll, "\pi_1"'] \arrow[ddl, "\pi_2"'] \arrow[ddd, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrr, "\pi_{n-1}"] \arrow[rrrr, dotted, two heads, "\text{surjective}"] & & & & \mathbf{Eq}(\nabla) \arrow[dl, dotted, shift left] \arrow[ddddddr, shift left] & \\
3F \arrow[rrrrrr, dashed, two heads, "1_F + \nabla"] \arrow[ddr, shift right, "\nabla + 1"'] & & & & & & 2F \arrow[ur, dotted, shift left] \arrow[ddddddr, dashed, shift left, "\nabla"'] \arrow[ddddddr, dashed, shift right] & & \\
 & & 3F \arrow[dll, shift left] \arrow[dd, shift left] & & & & & & \\
 & 2F \arrow[uul, shift right] \arrow[ur, shift left, "1+\nabla"] \arrow[ddddrrrrrr, dashed] & & 3F \arrow[dl, shift left] & & & & & \\
 & & 2F \arrow[uu, shift left] \arrow[ur, shift left] \arrow[dddrrrrr, dashed, "\nabla"'] & \ddots & & & & & \\
 & & & & 3F \arrow[dd, shift right, "\nabla + 1"'] & & & & \\
 & & & \ddots & & 3F \arrow[dl, shift left] \arrow[rrr, "\nabla + 1_F"] & & & 2F \arrow[uuuuuul, shift left] \arrow[dl, shift left, "\nabla"'] \\
 & & & & 2F \arrow[uu, shift right] \arrow[ur, shift left, "1+\nabla"] \arrow[rrr, "\nabla"'] & & \cdots & F \arrow[ur, shift left] &
\end{tikzcd}$$

- $\boldsymbol{n}$-permutable varieties: $\boldsymbol{F}$ - free algebra on one element

$$\nabla = \left(1_F \ 1_F\right) \colon 2F \to F$$

Avoiding coproducts

- $\mathbb{C}$ regular + (SP) $\Rightarrow$ $\mathbb{C}$ is an n-permutable category

Avoiding coproducts

Theory of unconditional exactness pps [JJ]

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$\downarrow$

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  $\mathbb{C}$ satisfies (P) $\Rightarrow$ $\mathbf{Lex}(\mathbb{C},\mathbf{Set})^{\mathbf{op}}$ satisfies (P)

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$$\Downarrow$$

$\mathbf{Lex}(\mathbb{C}, \mathbf{Set})^{\mathbf{op}}$ is n-permutable

Avoiding coproducts

Theory of unconditional exactness pps [JJ]

$$\begin{tikzcd} \mathbb{C} \text{ regular} + \text{(SP)} \arrow[r, Rightarrow, "\downarrow"] & \mathbb{C} \text{ is an } n\text{-permutable category} \end{tikzcd}$$

- Thm $\mathbb{C}$ small + lex, (P) unconditional exactness pp

  $\mathbb{C}$ satisfies (P) $\Rightarrow$ $\mathbf{Lex}(\mathbb{C}, \mathbf{Set})^{\mathbf{op}}$ satisfies (P)

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  -

$$\begin{tikzcd} \mathbb{C} \text{ regular} + \text{(SP)} \arrow[r, Rightarrow] & \mathbf{Lex}(\mathbb{C}, \mathbf{Set})^{\mathbf{op}} \text{ regular} + \text{(SP)} \arrow[d, Rightarrow] \\ \mathbb{C} \text{ is } n\text{-permutable} & \mathbf{Lex}(\mathbb{C}, \mathbf{Set})^{\mathbf{op}} \text{ is } n\text{-permutable} \arrow[l, Rightarrow, "RS\cdots = SR\cdots"'] \end{tikzcd}$$

Unconditional exactness properties

- *finite* diagram + finite (co)limits $\rightsquigarrow$ some map is an isomorphism

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- (SP)

$$\begin{tikzcd}
& & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"'] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\lambda"] & & & & & U \times_V W \arrow[dl, dashed] \arrow[ddddddr] \\
X \arrow[rrrrrrrrr, dashed, two heads, "\gamma"] \arrow[ddr, "f"'] & & & & & & & & & U \arrow[ur, dashed] \arrow[ddddddr, dashed, "h"'] \\
& & X \arrow[dl, "g"'] \arrow[ddr, "f"'] \\
& Y \arrow[uul, "s"'] \arrow[ur, "t"'] \arrow[ddddrrrrrrrrr, dashed, two heads, "\beta"] & & & X \arrow[dl, "g"'] \\
& & & Y \arrow[uul, "s"'] \arrow[ur, "t"'] \arrow[dddrrrrrrr, dashed, two heads, "\beta"] & & \ddots \\
& & \ddots & & & & X \arrow[ddr, "f"'] \\
& & & & & & & & X \arrow[dl, "g"'] \arrow[rrr, two heads, "\delta"] & \cdots & & W \arrow[dl, "k"'] \arrow[uuuuuul] \\
& & & & & & & Y \arrow[uul, "s"'] \arrow[ur, "t"'] \arrow[rrr, two heads, "\beta"'] & & & V \arrow[uuuuuul, dashed, "u"'] \arrow[ur, "w"']
\end{tikzcd}$$

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X \arrow[rrrrrrrrr, dashed, two heads, "\gamma"] \arrow[ddr, shift right, "f"'] & & & & & & & & & U \arrow[ur, dashed] \arrow[ddddddrr, dashed, shift right, "h"'] & & & \\
 & & X \arrow[dl, "g"'] \arrow[ddr, shift right, "f"'] & & & & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, "t"'] \arrow[ddddrrrrrrrrrr, dashed, "\beta"] & & & X \arrow[dl, "g"'] & & & & & & & & \\
 & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, "t"'] \arrow[dddrrrrrrrr, dashed, "\beta"] & & \ddots & & & & & & & \\
 & & & & \ddots & & X \arrow[ddr, shift right, "f"'] & & & & & & \\
 & & & & & & & & X \arrow[dl, "g"'] \arrow[rrrr, two heads, "\delta"] & & & & W \arrow[dl, "k"'] \arrow[uuuuuull, shift left] \\
 & & & & & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, "t"'] \arrow[rrrr, two heads, "\beta"'] & & & \cdots & V \arrow[ur, "w"'] \arrow[uuuuuull, dashed, shift right, "u"'] &
\end{tikzcd}$$

γ, δ regular epis $\Rightarrow$ λ regular epi

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 & & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\lambda"] & & & & & U\times_V W \arrow[dl, dashed] \arrow[ddddddr, shift left] & \\
X \arrow[ddr, shift right, "f"'] \arrow[rrrrrrrrr, dashed, two heads, "\gamma"] & & & & & & & & & U \arrow[ur, dashed] \arrow[ddddddr, dashed, shift right, "h"'] & & \\
 & & X \arrow[dl, shift right, "g"'] \arrow[ddr, shift right, "f"'] & & & & & & & & & \\
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 & & & & \ddots & & X \arrow[ddr, shift right, "f"'] & & & & & \\
 & & & & & & & & X \arrow[dl, shift right, "g"'] \arrow[rrr, two heads, "\delta"] & \cdots & & W \arrow[uuuuuul, shift left] \arrow[dl, shift right, "k"'] \\
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\end{tikzcd}$$

γ, δ regular epis $\Rightarrow$ λ regular epi   not "unconditional"

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 & & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\lambda"] & & & & & U \times_V W \arrow[dl, dashed] \arrow[ddddddr, shift left] & \\
X \arrow[ddr, shift right, "f"'] \arrow[rrrrrrrrr, dashed, two heads, "\gamma"] & & & & & & & & & U \arrow[ur, dashed] \arrow[ddddddr, dashed, shift right, "h"'] & & \\
 & & X \arrow[dl, shift right, "g"'] \arrow[ddr, shift right, "f"'] & & & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddrrrrrrrrr, dashed, "\beta"] & & & X \arrow[dl, shift right, "g"'] & & & & & & & \\
 & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddrrrrrrr, dashed, "\beta"] & \ddots & & & & & & & \\
 & & & & \ddots & & X \arrow[ddr, shift right, "f"'] & & & & & \\
 & & & & & & & & X \arrow[dl, shift right, "g"'] \arrow[rrr, two heads, "\delta"] & & & W \arrow[dl, shift right, "k"'] \arrow[uuuuuul, shift left] \\
 & & & & & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrr, two heads, "\beta"'] & & \cdots & V \arrow[ur, shift right, "w"'] \arrow[uuuuuul, dashed, shift right, "u"'] &
\end{tikzcd}$$

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 & & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"'] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"'] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrr, dotted, "\lambda'"] & & & I \times_V J \arrow[rr, dotted, "\lambda''"] \arrow[dl, dotted] \arrow[ddddddrr, shift right] & & U \times_V W \arrow[dl, dotted] \arrow[ddddddrr, shift right] & & \\
X \arrow[rrrrrrr, dashed, two heads, "p"] \arrow[ddr, shift right, "f"'] & & & & & & & I \arrow[ur, dashed] \arrow[rr, dashed, tail, "m"] \arrow[ddddddrrrr, dashed, shift right, "hm"'] & & U \arrow[ur, dashed] \arrow[ddddddrr, dotted, "h"'] & & & \\
 & & X \arrow[dl, shift right, "g"'] \arrow[ddr, shift right, "f"'] & & & & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddrrrrrrrrrr, dashed, "\beta"] & & & X \arrow[dl, shift right, "g"'] & & & & & & & & \\
 & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddrrrrrrrr, dashed, "\beta"] & & & & & & & & & \\
 & & & & \ddots & & X \arrow[ddr, shift right, "f"'] & & & & & & \\
 & & & & & & & & X \arrow[rr, two heads, "q"] \arrow[dl, shift right, "g"'] & & J \arrow[rr, tail, "n"] \arrow[dr, shift right] \arrow[uuuuuull, shift right] & & W \arrow[dl, shift right, "k"'] \arrow[uuuuuull, shift right] \\
 & & & & & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrrr, two heads, "\beta"'] & & & & V \arrow[uuuuuullll, dashed, shift right] \arrow[uuuuuull, dashed, "u"'] \arrow[ul, shift right] \arrow[ur, shift right, "w"'] &
\end{tikzcd}$$

λ' regular epi

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- *finite* diagram + finite (co)limits ⇝ some map is an isomorphism

- (SP')

$$\begin{tikzcd}
 & & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrr, dotted, "\lambda'"] & & & I \times_V J \arrow[rr, dotted, "\lambda''"] \arrow[dl, dotted] \arrow[ddddddrr, shift left] & & U \times_V W \arrow[dl, dotted] \arrow[ddddddrr, shift left] & & \\
X \arrow[rrrrrrr, dashed, two heads, "p"] \arrow[ddr, shift right, "f"'] & & & & & & & I \arrow[rr, dashed, tail, "m"] \arrow[ur, dashed] \arrow[ddddddrrrr, dashed, shift left] & & U \arrow[ur, dashed] \arrow[ddddddrr, dotted, "h"'] & & & \\
 & & X \arrow[dl, "g"'] \arrow[ddr, shift right, "f"'] & & & & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, "t"'] \arrow[ddddrrrrrrrrrr, dashed, "\beta"] & & & X \arrow[dl, "g"'] & & & & & & & & \\
 & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, "t"'] \arrow[dddrrrrrrrr, dashed, "\cdots\beta"] & & \ddots & & & & & & & \\
 & & & & \ddots & & X \arrow[ddr, shift right, "f"'] & & & & & & \\
 & & & & & & & & X \arrow[dl, "g"'] \arrow[rr, two heads, "q"] & & J \arrow[rr, tail, "n"] \arrow[dr, shift left] \arrow[uuuuuull, shift left] & & W \arrow[dl, "k"'] \arrow[uuuuuull, shift left] \\
 & & & & & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, "t"'] \arrow[rrrr, two heads, "\beta"'] & & & & V \arrow[ul, shift left] \arrow[ur, "w"'] \arrow[uuuuuullll, dashed, shift left, "hm"] \arrow[uuuuuull, dashed, "u"'] &
\end{tikzcd}$$

λ' regular epi

Unconditional exactness properties

- *finite* diagram + finite (co)limits $\rightsquigarrow$ some map is an isomorphism

- (SP') - UEP ✓

$$\begin{tikzcd}
 & & & & L \arrow[llll, "\pi_1"'] \arrow[ddll, "\pi_2"'] \arrow[ddd, "\pi_3"] \arrow[dddddrr, "\pi_{n-2}"] \arrow[ddddddrrrr, "\pi_{n-1}"] \arrow[rrr, dotted, "\lambda'"] & & & I\times_V J \arrow[d, dotted] \arrow[ddddddrrr, shift left] \arrow[rrr, dotted, "\lambda''"] & & & U\times_V W \arrow[d, dotted] \arrow[ddddddrr, shift left] & & \\
X \arrow[rrrrrrr, dashed, two heads, "p"] \arrow[ddr, shift right, "f"'] & & & & & & & I \arrow[u, dashed, shift right] \arrow[rrr, dashed, tail, "m"] \arrow[ddddddrrrr, dashed, shift left, "hm"] & & & U \arrow[u, dashed, shift right] \arrow[ddddddr, dotted, "h"'] & & \\
 & & X \arrow[dl, shift right, "g"'] \arrow[ddr, shift right, "f"'] & & & & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddrrrrrrrrrr, dashed, two heads, "\beta"] & & & X \arrow[dl, shift right, "g"'] & & & & & & & & \\
 & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddrrrrrrrr, dashed, two heads, "\beta"] & & \ddots & & & & & & & \\
 & & & & \ddots & & X \arrow[ddr, shift right, "f"'] & & & & & & \\
 & & & & & & & & X \arrow[dl, shift right, "g"'] \arrow[rr, two heads, "q"] & & J \arrow[uuuuuulll, shift left] \arrow[rr, tail, "n"] \arrow[dr, shift left] & & W \arrow[uuuuuull, shift left] \arrow[dl, shift left, "k"] \\
 & & & & & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrrr, two heads, "\beta"'] & & & & V \arrow[uuuuuullll, dashed, shift left] \arrow[uuuuuul, dashed, "u"'] \arrow[ul, shift left] \arrow[ur, shift left, "w"'] &
\end{tikzcd}$$

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 & & & L \arrow[dlll, "\pi_1"'] \arrow[ddll, "\pi_2"] \arrow[dddl, "\pi_3"] \arrow[ddddd, "\pi_{n-2}"] \arrow[ddddddr, "\pi_{n-1}"] \arrow[rr, dotted, "\lambda'"] & & I \times_V J \arrow[d, dotted] \arrow[ddddddrr] \arrow[rr, dotted, "\lambda''\quad \text{mono}"] & & U \times_V W \arrow[d, dotted] \arrow[ddddddr] & \\
X \arrow[rrrrr, dashed, two heads, "p"] \arrow[dd, shift right, "f"'] & & & & & I \arrow[u, dashed] \arrow[rr, dashed, tail, "m"] \arrow[ddddddrr, dashed, "hm"] & & U \arrow[u, dashed] \arrow[dddddd, dotted, "h"'] & \\
 & X \arrow[dl, shift right, "g"'] \arrow[dd, shift right, "f"'] & & & & & & & \\
Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddrrrrrrr, dashed, "\beta"] & & X \arrow[dl, shift right, "g"'] & & & & & & \\
 & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddrrrrrr, dashed, "\beta"] & \ddots & & & & & & \\
 & & \ddots & X \arrow[dd, shift right, "f"'] & & & & & \\
 & & & & X \arrow[dl, shift right, "g"'] \arrow[rrr, two heads, "q"] & & & J \arrow[uuuuuull] \arrow[r, tail, "n"] \arrow[d, shift right] & W \arrow[uuuuuul] \arrow[dl, shift right, "k"'] \\
 & & & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrrr, two heads, "\beta"'] & & & \cdots & V \arrow[uuuuuull, dashed] \arrow[uuuuuu, dashed, "u"'] \arrow[u, shift right] \arrow[ur, shift right, "w"'] &
\end{tikzcd}$$

λ' regular epi

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$$\begin{tikzcd}
 & & & & \cdot \arrow[d, tail] \arrow[drrr, tail, "\mathrm{im}(\lambda)"] & & & & \\
 & & & L \arrow[ur, two heads] \arrow[r, dotted, "\lambda'"] \arrow[dlll, "\pi_1"'] \arrow[ddll, "\pi_2"] \arrow[dddl, "\pi_3"] \arrow[ddddd, "\pi_{n-2}"] \arrow[ddddddr, "\pi_{n-1}"] & I \times_V J \arrow[rrr, dotted, "\lambda''\quad\text{mono}"] \arrow[d, dotted, shift right] \arrow[ddddddrrr, shift left] & & & U \times_V W \arrow[d, dotted, shift right] \arrow[ddddddr, shift left] & \\
X \arrow[rrrr, dashed, two heads, "p"] \arrow[dd, shift right, "f"'] & & & & I \arrow[u, dashed, shift right] \arrow[rrr, dashed, tail, "m"] \arrow[ddddddrrr, dashed, shift left, "hm"] & & & U \arrow[u, dashed, shift right] \arrow[dddddd, dotted, shift right, "h"'] & \\
 & X \arrow[dl, shift right, "g"'] \arrow[dd, shift right, "f"'] & & & & & & & \\
Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddrrrrrrr, dashed, "\beta"] & & X \arrow[dl, shift right, "g"'] & & & & & & \\
 & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddrrrrrr, dashed, "\beta"] & & & & & & & \\
 & & \ddots & X \arrow[dd, shift right, "f"'] & & & & & \\
 & & & & X \arrow[rrr, "q"] \arrow[dl, shift right, "g"'] & & & J \arrow[r, tail, "n"] \arrow[uuuuuulll, shift left] \arrow[d, shift right] & W \arrow[uuuuuul, shift left] \arrow[dl, shift right, "k"'] \\
 & & & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrrr, "\beta"'] & & & & V \arrow[u, shift right] \arrow[ur, shift right, "w"'] \arrow[uuuuuulll, dashed] \arrow[uuuuuu, dashed, shift right, "u"'] &
\end{tikzcd}$$

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$$\begin{tikzcd}
 & & & & & \cdot \arrow[d, tail, "\ncong"] \arrow[rrd, tail, "\mathbf{im}(\lambda)"] & & & & \\
 & & & L \arrow[rru] \arrow[rr, dotted, "\lambda'"] \arrow[llld, "\pi_1"'] \arrow[lldd, "\pi_2"] \arrow[lddd, "\pi_3"] \arrow[ddddd, "\pi_{n-2}"] \arrow[rrdddddd, "\pi_{n-1}"] & & I \times_V J \arrow[rr, dotted, tail, "\lambda'' \quad \text{mono}"] \arrow[d, dotted, shift right] \arrow[rrdddddd, shift left] & & U \times_V W \arrow[d, dotted, shift right] \arrow[rrdddddd, shift left] & & \\
X \arrow[rrrrr, dashed, two heads, "p"] \arrow[dd, shift right, "f"'] & & & & & I \arrow[u, dashed, shift right] \arrow[rr, dashed, tail, "m"] \arrow[rrrdddddd, dashed, shift left, "hm"] & & U \arrow[u, dashed, shift right] \arrow[rdddddd, dotted, "h"] & & \\
 & X \arrow[ld, shift right, "g"'] \arrow[dd, shift right, "f"'] & & & & & & & & \\
Y \arrow[uu, shift right, "s"'] \arrow[ru, shift right, "t"'] \arrow[rrrrrrrrdddd, dashed, "\beta"] & & X \arrow[ld, shift right, "g"'] & & & & & & & \\
 & Y \arrow[uu, shift right, "s"'] \arrow[ru, shift right, "t"'] \arrow[rrrrrrrddd, dashed, "\beta"] & \ddots & & & & & & & \\
 & & \ddots & X \arrow[rdd, shift right, "f"'] & & & & & & \\
 & & & & & X \arrow[ld, shift right, "g"'] \arrow[rr, "q"] & & J \arrow[rr, tail, "n"] \arrow[rd, shift left] \arrow[lluuuuuu, shift left] & & W \arrow[ld, shift right, "k"'] \arrow[lluuuuuu, shift left] \\
 & & & & Y \arrow[luu, shift right, "s"'] \arrow[ru, shift right, "t"'] \arrow[rrrr, "\beta"'] & & & \cdots \beta & V \arrow[lu, shift left] \arrow[ru, shift right, "w"'] \arrow[llluuuuuu, dashed, shift left] \arrow[luuuuuu, dashed, "u"] &
\end{tikzcd}$$

λ' regular epi

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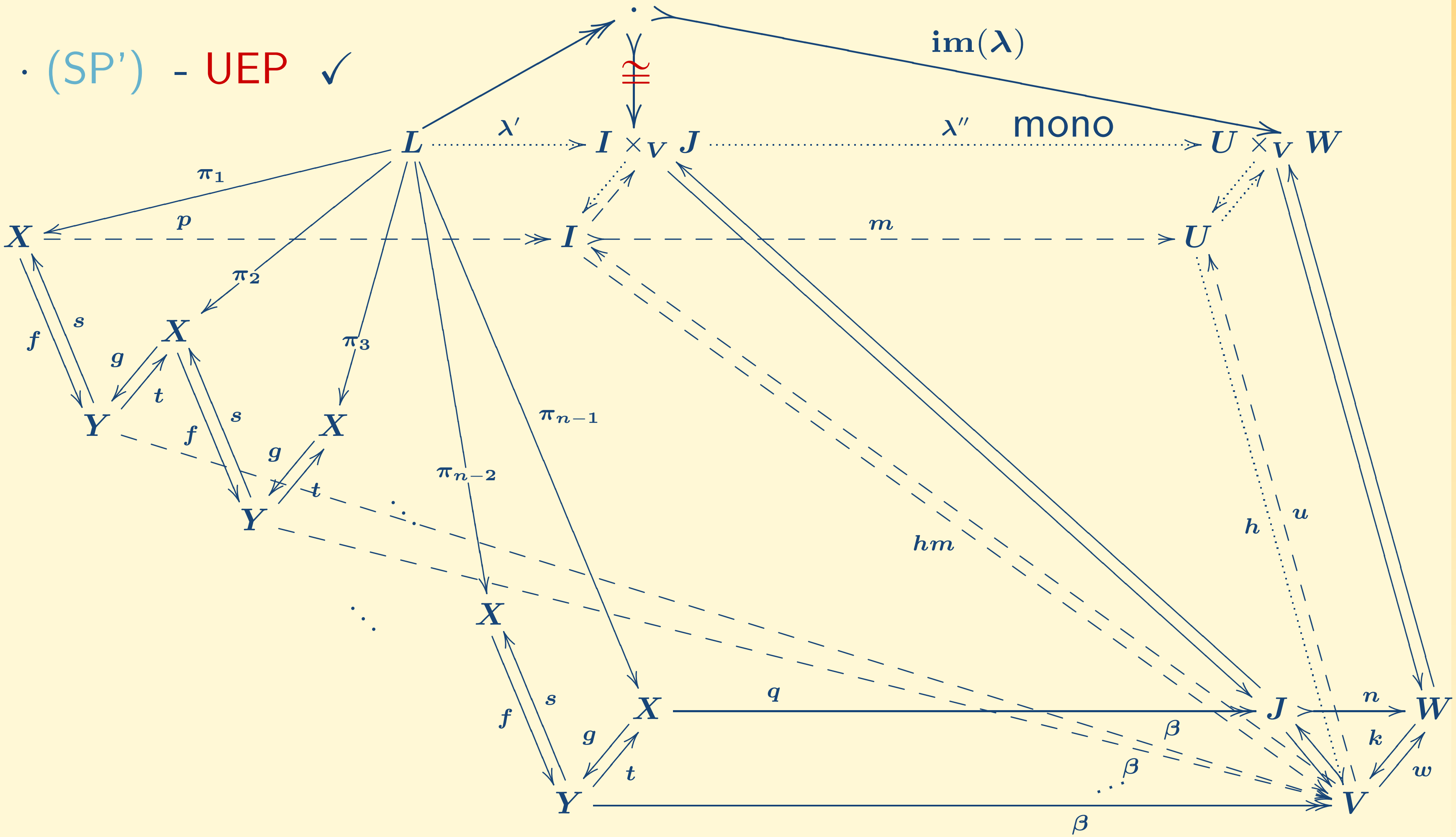

λ' regular epi

The finite issue

- *finite*: the category generated by a finite conditional graph is finite

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$\mathscr{G}$ finite conditional graph $\Rightarrow$ $\mathbf{Path}(\mathscr{G})$ finite category

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$$\mathscr{G} \text{ finite conditional graph } \Rightarrow \mathbf{Path}(\mathscr{G}) \text{ finite category}$$

- $\mathscr{G}$:

$$\begin{tikzcd}
& X \arrow[r, bend left, "f"] \arrow[r, shift left, "g"] \arrow[d, "\delta"'] \arrow[ddl, bend right, "\gamma"'] & Y \arrow[l, shift left, "s"] \arrow[l, bend left, "t"] \arrow[d, bend right, "\beta"'] \\
& W \arrow[r, shift left, "k"] & V \arrow[l, shift left, "w"] \arrow[u, "i"'] \arrow[u, bend right, "i'"'] \arrow[dll, bend left, "u"] \\
U \arrow[urr, bend left, "h"] & &
\end{tikzcd}$$

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 & W \arrow[r, "k"] & V \arrow[l, "w"] \arrow[u, "i"'] \arrow[u, bend right, "i'"'] \arrow[dll, bend left, "u"] \\
U \arrow[urr, bend right, "h"] & &
\end{tikzcd}$$

$$fs = gt = 1_Y$$
$$\beta i = \beta i' = kw = hu = 1_V$$
$$\beta g = \beta f = h\gamma = k\delta$$
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 & X \arrow[r, bend left, "f"] \arrow[r, shift left, "g"] \arrow[d, "\delta"'] \arrow[ddl, bend right, "\gamma"'] & Y \arrow[l, shift left, "s"] \arrow[l, bend left, "t"] \arrow[d, bend right, "\beta"'] \\
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(SP'): $\mathbf{Path}(\mathscr{G})$ + finite (co)limits

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$$\begin{tikzcd}
& X \arrow[r, bend left, "f"] \arrow[r, "g"] \arrow[d, "\delta"'] \arrow[ddl, bend right, "\gamma"'] & Y \arrow[l, "s"] \arrow[l, bend left, "t"] \arrow[d, bend right, "\beta"'] \\
& W \arrow[r, "k"] & V \arrow[l, "w"] \arrow[u, "i"'] \arrow[u, bend right, "i'"'] \arrow[dll, "u"] \\
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(SP'): $\mathbf{Path}(\mathscr{G})$ + finite (co)limits

- Algorithm: look at subgraphs of $\mathscr{G}$ by eliminating objects

The algorithm

- $\mathscr{G}$:

$$\begin{tikzcd}
 & X \arrow[r, bend left, "f"] \arrow[r, shift left, "g"] \arrow[d, "\delta"'] \arrow[ddl, bend right, "\gamma"'] & Y \arrow[l, shift left, "s"] \arrow[l, bend left=50, "t"] \arrow[d, bend right, "\beta"'] \\
 & W \arrow[r, shift left, "k"] & V \arrow[l, shift left, "w"] \arrow[u, "i"'] \arrow[u, bend right, "i'"'] \arrow[dll, bend left=10, shift left, "u"] \\
U \arrow[urr, bend right=10, shift left, "h"] & &
\end{tikzcd}$$

$$fs = gt = 1_Y$$
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- $\mathbf{Path}(\mathscr{G})$ finite iff $\mathbf{Path}(\mathscr{G}\backslash\{U\})$ finite

The algorithm

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$$\begin{tikzcd}
& X \arrow[r, bend left, "f"] \arrow[r, "g"] \arrow[d, "\delta"'] \arrow[ddl, bend right, "\gamma"'] & Y \arrow[l, "s"] \arrow[l, bend left, "t"] \arrow[d, bend right, "\beta"'] \\
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U \arrow[urr, bend right, "h"] & &
\end{tikzcd}$$

$$fs = gt = 1_Y$$
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$$\begin{tikzcd}
\cdot \arrow[rr] \arrow[dr] & & U \arrow[rr] & & \cdot \\
& \cdot \arrow[rr, dotted, no head] & & \cdot \arrow[ur] &
\end{tikzcd}$$

The algorithm

- $\mathscr{G}$:

$$\begin{tikzcd}
 & X \arrow[r, bend left, "f"] \arrow[r, "g"] \arrow[d, "\delta"'] \arrow[ddl, bend right, "\gamma"'] & Y \arrow[l, "s"] \arrow[l, bend left, "t"] \arrow[d, bend right, "\beta"'] \\
 & W \arrow[r, "k"] & V \arrow[u, "i"'] \arrow[u, bend right, "i'"'] \arrow[l, "w"] \arrow[dll, bend left, "u"] \\
U \arrow[urr, bend right, "h"] & &
\end{tikzcd}$$

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$$\begin{tikzcd}
\cdot \arrow[r] \arrow[dr] & U \arrow[r] & \cdot \\
 & \cdots \arrow[ur] &
\end{tikzcd}$$

$$h\gamma = \beta f, \; hu = 1_V$$

The algorithm

- $\mathscr{G}$:

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, "g"] \arrow[d, "\delta"'] & Y \arrow[l, "s"] \arrow[l, bend left, "t"] \arrow[d, bend right, "\beta"'] \\
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$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, "g"] \arrow[d, "\delta"'] & Y \arrow[l, "s"] \arrow[l, bend left, "t"] \arrow[d, bend right, "\beta"'] \\
W \arrow[r, shift left, "k"] & V \arrow[l, shift left, "w"] \arrow[u, "i"'] \arrow[u, bend right, "i'"']
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  iff $\mathbf{Path}(\mathscr{G}\backslash\{U, W\})$ finite

The algorithm

- $\mathscr{G}$:

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, shift left, "g"] & Y \arrow[l, shift left, "s"] \arrow[l, bend left, "t"] \arrow[d, bend right, "\beta"'] \\
 & V \arrow[u, "i"'] \arrow[u, bend right, "i'"']
\end{tikzcd}$$

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$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, shift left, "g"] & Y \arrow[l, shift left, "s"] \arrow[l, bend left, "t"] \arrow[d, bend right, "\beta"'] \\
& V \arrow[u, "i"'] \arrow[u, bend right, "i'"']
\end{tikzcd}$$

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The algorithm

- $\mathscr{G}$:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, "g"] & Y \arrow[l, "s"] \arrow[l, bend left, "t"] \end{tikzcd}$$

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$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, "g"] & Y \arrow[l, "s"] \arrow[l, bend left, "t"] \end{tikzcd}$$

$$fs = gt = 1_Y \quad \checkmark$$

$$ft = i\beta, \ gs = i'\beta$$

- $\mathbf{Path}(\mathscr{G})$ finite iff $\mathbf{Path}(\mathscr{G} \backslash \{U\})$ finite

  iff $\mathbf{Path}(\mathscr{G} \backslash \{U, W\})$ finite

  iff $\mathbf{Path}(\mathscr{G} \backslash \{U, W, V\})$ finite

The algorithm

- $\mathscr{G}$:

$$\begin{tikzcd}
X \arrow[r, bend left=40, "f"] \arrow[r, shift left, "g"] & Y \arrow[l, shift left, "s"] \arrow[l, bend left=40, "t"]
\end{tikzcd}$$

$$fs = gt = 1_Y \ \checkmark$$

$$y = ft = i\beta, \ gs = i'\beta = y'$$

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iff $\mathbf{Path}(\mathscr{G}\backslash\{U, W, V\})$ finite

The algorithm

- $\mathscr{G}$:

$$\begin{tikzcd} Y \arrow[loop left, "y"] \arrow[loop right, "y'"] \end{tikzcd}$$

$$fs = gt = 1_Y \quad \checkmark$$

$$y = ft = i\beta,\ gs = i'\beta = y'$$

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The algorithm

· $\mathscr{G}$:

$$\begin{tikzcd} \boldsymbol{Y} \arrow[loop left, "y"] \arrow[loop right, "y'"] \end{tikzcd}$$

$$fs = gt = 1_Y \quad ✓$$

$$y = ft = i\beta,\ gs = i'\beta = y'$$

$$yy = ftft = i\beta i\beta = i\beta = ft = y$$

· $\mathbf{Path}(\mathscr{G})$ finite iff $\mathbf{Path}(\mathscr{G}\backslash\{U\})$ finite

iff $\mathbf{Path}(\mathscr{G}\backslash\{U, W\})$ finite

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$$fs = gt = 1_Y \quad \checkmark$$

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$$yy = ftft = i\beta i\beta = i\beta = ft = y$$

$$yy' = y,\ y'y = y',\ y'y' = y'$$

- $\mathbf{Path}(\mathscr{G})$ finite iff $\mathbf{Path}(\mathscr{G}\backslash\{U\})$ finite

  iff $\mathbf{Path}(\mathscr{G}\backslash\{U, W\})$ finite

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The algorithm

- $\mathscr{G}$:

$$\begin{tikzcd} \boldsymbol{Y} \arrow[loop left, "y"] \arrow[loop right, "y'"] \end{tikzcd}$$

$$fs = gt = 1_Y \quad \checkmark$$

$$y = ft = i\beta,\ gs = i'\beta = y'$$

$$yy = ftft = i\beta i\beta = i\beta = ft = y$$

$$yy' = y,\ y'y = y',\ y'y' = y'$$

$\Rightarrow$ $\mathbf{Path}(\mathscr{G})$ is finite

- $\mathbf{Path}(\mathscr{G})$ finite iff $\mathbf{Path}(\mathscr{G}\backslash\{U\})$ finite

iff $\mathbf{Path}(\mathscr{G}\backslash\{U, W\})$ finite

iff $\mathbf{Path}(\mathscr{G}\backslash\{U, W, V\})$ finite

Another stability property

Another stability property

- **Thm** $\mathbb{C}$ regular, $n \geqslant 3$. $\mathbb{C}$ is an n-permutable cat iff for any

$$\begin{tikzcd}
 & & & & L \arrow[rrrrr, dotted, two heads, "\lambda"] \arrow[dllll, "\pi_1"'] \arrow[ddll, "\pi_2"] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrr, "\pi_{n-1}"] & & & & & U\times_V W \arrow[dl, dashed] \arrow[ddddddr, shift left] & \\
X \arrow[rrrrrrrr, dashed, two heads, "\gamma"] \arrow[ddr, shift right, "f"'] & & & & & & & & U \arrow[ur, dashed] \arrow[ddddddr, dashed, shift right, "h"'] & & \\
 & & X \arrow[dl, shift right, "g"'] \arrow[dd, shift right, "f"'] & & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddrrrrrrrr, dashed, two heads] & & X \arrow[dl, shift right, "g"'] & & & & & & & \\
 & & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddrrrrrrr, dashed, two heads, "\beta"] & & \ddots & & & & & & \\
 & & & & \ddots & X \arrow[dd, shift right, "f"'] & & & & & \\
 & & & & & & X \arrow[dl, shift right, "g"'] \arrow[rrrr, two heads, "\delta"] & & & & W \arrow[uuuuuul, shift left] \arrow[dl, shift right, "k"'] \\
 & & & & & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrrr, two heads, "\beta"'] & & & \cdots \;\beta & V \arrow[uuuuuul, dashed, shift right, "u"'] \arrow[ur, shift right, "w"'] &
\end{tikzcd}$$

Another stability property

- Thm $\mathbb{C}$ regular, $n \geqslant 3$. $\mathbb{C}$ is an n-permutable cat iff for any

$$\begin{tikzcd}
 & & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"'] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\lambda"] & & & & & U \times_V W \arrow[dl, dashed] \arrow[ddddddrr, shift left] & & \\
X \arrow[ddr, shift right, "f"'] \arrow[rrrrrrrrr, dashed, two heads, "\gamma"] & & & & & & & & & U \arrow[ur, dashed] \arrow[ddddddrr, dashed, shift right, "h"'] & & & \\
 & & X \arrow[dl, shift right, "g"'] \arrow[ddr, shift right, "f"'] & & & & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrrrrrrrrrdddd, dashed, two heads, "\beta"] & & & X \arrow[dl, shift right, "g"'] & & & & & & & & \\
 & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrrrrrrrddd, dashed, two heads, "\beta"] & \ddots & & & & & & & & \\
 & & & & \ddots & & X \arrow[ddr, shift right, "f"'] & & & & & & \\
 & & & & & & & & X \arrow[dl, shift right, "g"'] \arrow[rrrr, two heads, "\delta"] & & & & W \arrow[dl, shift right, "k"'] \arrow[uuuuuull, shift left] \\
 & & & & & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrrr, two heads, "\beta"'] & & & \cdots & V \arrow[ur, shift right, "w"'] \arrow[uuuuuull, dashed, shift right, "u"'] &
\end{tikzcd}$$

γ, δ regular epis $\Rightarrow$ λ regular epi

Another stability property

- **Thm** $\mathbb{C}$ regular, $n \geqslant 3$. $\mathbb{C}$ is an n-permutable cat iff for any

$$\begin{tikzcd}
 & & & L \arrow[lll, "\pi_1"'] \arrow[ddl, "\pi_2"] \arrow[ddd, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\lambda"] & & & & & U \times_V W \arrow[dl, dashed] \arrow[dddddd, shift left] \\
X \arrow[rrrrrrr, dashed, two heads, "\gamma"] \arrow[ddr, shift right, "f"'] & & & & & & & U \arrow[ur, dashed] \arrow[ddddddr, dashed, shift right, "h"'] & \\
 & & X \arrow[dl, shift right, "g"'] \arrow[dd, shift right, "f"'] & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddrrrrrrr, dashed] & & X \arrow[dl, shift right, "g"'] & & & & & \\
 & & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddrrrrrr, dashed, "\beta"] & \ddots & & & & & \\
 & & \ddots & & X \arrow[dd, shift right, "f"'] \arrow[ddrrrr, dashed, "\beta"] & & & & \\
 & & & & & X \arrow[dl, shift right, "g"'] \arrow[rrr, two heads, "\delta"] & & & W \arrow[uuuuuu, shift left] \arrow[d, shift right, "k"'] \\
 & & & & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrrr, two heads, "\beta"'] & & & \cdots & V \arrow[uuuuuul, dashed, shift right, "u"'] \arrow[u, shift right, "w"']
\end{tikzcd}$$

γ, δ regular epis $\Rightarrow$ λ regular epi

$\rightsquigarrow$ recover the ternary terms

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 & & & L \arrow[dlll, "\pi_1"'] \arrow[ddll, "\pi_2"'] \arrow[dddl, "\pi_3"] \arrow[ddddd, "\pi_{n-2}"] \arrow[ddddddr, "\pi_{n-1}"] \arrow[rrrr, dotted, two heads, "\lambda"] & & & & U \times_V W \arrow[dl, shift left] \arrow[dddddd, shift left] \\
X \arrow[dd, shift right, "f"'] \arrow[rrrrrr, dashed, two heads, "\gamma"] & & & & & & U \arrow[ur, shift left] \arrow[dddddd, dashed, shift right, "h"'] & \\
 & X \arrow[dl, shift right, "g"'] \arrow[dd, shift right, "f"'] & & & & & & \\
Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddrrrrrr, dashed, two heads] & & X \arrow[dl, shift right, "g"'] & & & & & \\
 & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddrrrrr, dashed, two heads, "\beta"] & \ddots & & & & & \\
 & & \ddots & X \arrow[dd, shift right, "f"'] & & & & \\
 & & & & X \arrow[dl, shift right, "g"'] \arrow[rrr, two heads, "\delta"] & & & W \arrow[dl, shift right, "k"'] \arrow[uuuuuu, shift left] \\
 & & & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrr, two heads, "\beta"'] & & \cdots\ \beta & V \arrow[ur, shift right, "w"'] \arrow[uuuuuu, dashed, shift right, "u"'] &
\end{tikzcd}$$

γ, δ regular epis $\Rightarrow$ λ regular epi

$\rightsquigarrow$ recover the ternary terms

$\boxed{\text{What about the } (n+1)\text{-ary terms?}}$

Another stability property

- Thm $\mathbb{C}$ regular, $n \geqslant 3$. $\mathbb{C}$ is an n-permutable cat iff for any

n odd

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$$\begin{tikzcd}
 & & & & & L \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"] \arrow[dddl, "\pi_3"] \arrow[dddddr, "\pi_{n-2}"] \arrow[ddddddrrr, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\lambda"] & & & & & \mathrm{Eq}(h) \arrow[dl, dotted] \arrow[ddddddr] & \\
X \arrow[rrrrrrrrr, dashed, "g"] \arrow[ddr, shift right, "f"'] & & & & & & & & & Z \arrow[ddddddr, dashed, "h"] & & \\
 & & X \arrow[dl, shift right, "f"'] \arrow[ddr, shift right, "g"'] & & & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "s"'] & & & X \arrow[dl, shift right, "g"'] & & & & & & & \\
 & & & Z \arrow[uul, shift right, "t"'] \arrow[ur, shift right, "t"'] & & \ddots & & & & & & \\
 & & & & \ddots & & X \arrow[ddr, shift right, "f"'] & & & & & \\
 & & & & & & & & X \arrow[dl, shift right, "f"'] \arrow[rrr, "g"] & & & Z \arrow[dl, "h"] \\
 & & & & & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "s"'] \arrow[rrr, two heads, "k"'] & & & V &
\end{tikzcd}$$

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$$\begin{tikzcd}
& & & & & L \arrow[rrrrrrr, dotted, two heads, "\lambda"] \arrow[dlllll, "\pi_1"'] \arrow[ddlll, "\pi_2"] \arrow[dddl, "\pi_3"] \arrow[dddddrr, "\pi_{n-2}"] \arrow[ddddddrrrr, "\pi_{n-1}"] & & & & & & & \mathbf{Eq}(h) \arrow[dl, dotted] \arrow[dddddd] \\
X \arrow[rrrrrrrrrrr, dashed, two heads, "g"] \arrow[ddr, shift right, "f"'] & & & & & & & & & & & Z \arrow[dddddd, dashed, "h"] & \\
& & X \arrow[dl, shift right, "f"'] \arrow[ddr, shift right, "g"'] & & & & & & & & & & \\
& Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "s"'] & & & X \arrow[dl, shift right, "g"'] & & & & & & & & \\
& & & Z \arrow[uul, shift right, "t"'] \arrow[ur, shift right, "t"'] & & & \ddots & & & & & & \\
& & & & & \ddots & & X \arrow[ddr, shift right, "f"'] & & & & & \\
& & & & & & & & & X \arrow[rrr, two heads, "g"] \arrow[dl, shift right, "f"'] & & & Z \arrow[dl, "h"] \\
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h/k is split + extra commutativity conditions

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X \arrow[ddr, shift right, "f"'] \arrow[rrrrrrrrr, dashed, two heads, "g"] & & & & & & & & & Z \arrow[ddddddrr, dashed, "h"] & & & \\
 & & X \arrow[dl, shift right, "f"'] \arrow[ddr, shift right, "g"'] & & & & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "s"'] & & & X \arrow[dl, shift right, "g"'] & & & & & & & & \\
 & & & Z \arrow[uul, shift right, "t"'] \arrow[ur, shift right, "t"'] & & \ddots & & & & & & & \\
 & & & & \ddots & & X \arrow[ddr, shift right, "f"'] & & & & & & \\
 & & & & & & & & X \arrow[dl, shift right, "f"'] \arrow[rrrr, two heads, "g"] & & & & Z \arrow[dl, "h"] \\
 & & & & & & & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "s"'] \arrow[rrrr, two heads, "k"'] & & & & V &
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X \arrow[rrrrrrrrrr, dashed, "g"] \arrow[ddr, shift right, "f"'] & & & & & & & & & & Z \arrow[dddddd, dashed, "h"] & \\
 & & X \arrow[dl, shift right, "f"'] \arrow[ddr, shift right, "g"'] & & & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "s"'] & & & X \arrow[dl, shift right, "g"'] & & & & & & & \\
 & & & Z \arrow[uul, shift right, "t"'] \arrow[ur, shift right, "t"'] & & \ddots & & & & & & \\
 & & & & \ddots & & X \arrow[ddr, shift right, "f"'] & & & & & \\
 & & & & & & & & X \arrow[dl, shift right, "f"'] \arrow[rrr, "g"] & & & Z \arrow[dl, "h"] \\
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n odd

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$\rightsquigarrow$ recover the $(n+1)$-ary terms

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$$\begin{tikzcd}
 & & & & & L \arrow[llllld, "\pi_1"'] \arrow[llldd, "\pi_2"] \arrow[lddd, "\pi_3"] \arrow[rdddddd, "\pi_{n-2}"] \arrow[rrrdddddd, "\pi_{n-1}"] \arrow[rrrrr, dotted, two heads, "\lambda"] & & & & & Z \times_V Y \arrow[ld, dotted, two heads] \arrow[rrdddddd] & & \\
X \arrow[rrrrrrrrr, dashed, two heads, "g"] \arrow[rdd, shift right, "f"'] & & & & & & & & & Z \arrow[rrdddddd, dashed, "h"] & & & \\
 & & X \arrow[ld, shift right, "f"'] \arrow[rdd, shift right, "g"'] & & & & & & & & & & \\
 & Y \arrow[luu, shift right, "s"'] \arrow[ru, shift right, "s"'] & & & X \arrow[ld, shift right, "g"'] & & & & & & & & \\
 & & & Z \arrow[luu, shift right, "t"'] \arrow[ru, shift right, "t"'] & & \ddots & & & & & & & \\
 & & & & & \ddots & X \arrow[rdd, shift right, "g"'] & & & & & & \\
 & & & & & & & & X \arrow[ld, shift right, "g"'] \arrow[rrrr, "f"] & & & & Y \arrow[ld, two heads, "k"] \\
 & & & & & & & Z \arrow[luu, shift right, "t"'] \arrow[ru, shift right, "t"'] \arrow[rrrr, two heads, "h"'] & & & & V &
\end{tikzcd}$$

n even

h/k is split + extra commutativity conditions $\Rightarrow$ λ regular epi

$\rightsquigarrow$ recover the $(n+1)$-ary terms