Symmetric lenses and universality

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Outline

- Lenses: symmetric and asymmetric
- Cospans and symmetric lenses
- Universality and compatibility

Lens

- Consider model domains $\mathbf{X}, \mathbf{Y}$... of *model states*
- Model states X, Y might be:
  elements of a set, of an order, objects of a category
- *Synchronization data* (various encodings) specifies *consistency* between an $\mathbf{X}$ state and a $\mathbf{Y}$ state
- Lens $L : \mathbf{X} \longrightarrow \mathbf{Y}$ is an example of a so-called *Bidirectional Transformation (BX)* and has both:
  - *synchronization data* and
  - *consistency restoration* or *re-synchronization* operator(s) responding to state change.

Lens

- *Symmetric* and *asymmetric* cases arise with different, but related, motivation...
- Asymmetric: Only one non-trivial restoration operator returns **X** (global) state change after **Y** (local) change: the motivating example: database view updates
- Symmetric: Concurrent models with bidirectional (two-way) re-synchronization: **X** and **Y** peers
motivating example: database interoperation

In more detail...

Symmetric lens

Consistency data (synchronization) for states X in $\mathbf{X}$ and Y in $\mathbf{Y}$ denoted by $R : X \leftrightarrow Y$.

Suppose X synchronized with Y by $R : X \leftrightarrow Y$,
then given an *update* from state X (with target X', say)

a *symmetric lens* delivers an update to Y (target Y', say)
and, re-synchronization $R' : X' \leftrightarrow Y'$.

$$X \xleftrightarrow{\;R\;} Y$$

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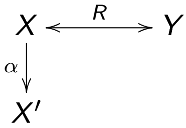

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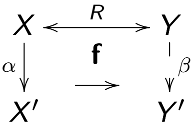

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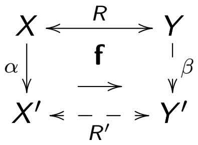
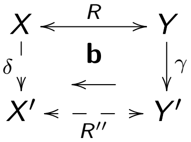


Symmetric lens

Symmetrically, suppose $R : X \leftrightarrow Y$, then given an *update* from Y (with target Y')

symmetric lens delivers update of X in $\mathbf{X}$ *and*, *re-synchronization* $R'' : X' \leftrightarrow Y'$.



- Considered by Hoffman, Pierce, Wagner for $\mathbf{X}, \mathbf{Y}$... sets
- More recently Diskin et al. for $\mathbf{X}, \mathbf{Y}$... categories
- Also studied by J & R

Symmetric lens

Formally, taking categories $\mathbf{X}, \mathbf{Y}$ for model domains:

A symmetric lens $L = (\delta_{\mathbf{X}}, \delta_{\mathbf{Y}}, \mathbf{f}, \mathbf{b})$ from $\mathbf{X}$ to $\mathbf{Y}$ has a span of *sets*

$$\delta_{\mathbf{X}} : |\mathbf{X}| \longleftarrow \mathbf{R_{XY}} \longrightarrow |\mathbf{Y}| : \delta_{\mathbf{Y}}$$

where elements of $\mathbf{R_{XY}}$ are denoted $R : X \leftrightarrow Y$ and *forward* and *backward propagations* $\mathbf{f}, \mathbf{b}$ denoted

$$\begin{array}{ccc} X & \overset{R}{\longleftrightarrow} & Y \\ {\scriptstyle\alpha}\downarrow & \mathbf{f} \ \longrightarrow & \downarrow{\scriptstyle\beta} \\ X' & \underset{R'}{\leftarrow - - \rightarrow} & Y' \end{array} \qquad \begin{array}{ccc} X & \overset{R}{\longleftrightarrow} & Y \\ {\scriptstyle\delta}\downarrow & \mathbf{b} \ \longleftarrow & \downarrow{\scriptstyle\gamma} \\ X' & \underset{R''}{\leftarrow - - \rightarrow} & Y' \end{array}$$

where $\mathbf{f}(\alpha, R) = (\beta, R')$ and $\mathbf{b}(\gamma, R) = (\delta, R'')$ and both propagations respect identities and composition.

Aside: $\mathbf{f}, \mathbf{b}$ are *Mealy morphisms* in **cat** (noted by Bob Paré)

Examples: To come, but first...

Asymmetric lens: Background

Arose as strategy for solving the database View Update Problem, actually defined well before symmetric lenses.

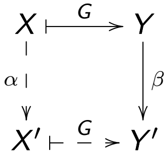
- Defined equationally by Pierce et al when $\mathbf{X}, \mathbf{Y}$ are sets
- (Equivalent) axioms from Hegner when $\mathbf{X}, \mathbf{Y}$ are orders
- J & R considered for $\mathbf{X}, \mathbf{Y}$ categories, then
  - defined asymmetric lens *in* category $\mathcal{C}$ with finite products
  - characterized lens as algebra for a monad on $\mathcal{C}/\mathbf{Y}$
  - generalized to a categorical version (c-lenses, to come).
- Diskin et al. defined (related) *asymmetric d-lenses*

Also arose in considering "abstract models of storage" (where there is a similar update problem)

Asymmetric lens: Motivation

Database *view* considered a *get* process $G : \mathbf{X} \longrightarrow \mathbf{Y}$
full database states $\mathbf{X}$ *to* view states $\mathbf{Y}$.

For global state X *synched* with view state $Y = GX$:
when can update to Y, e.g. formal insertion β
lift through G to global update α, *and*
compatibly – meaning $\beta = G(\alpha)$?
This is (an instance of) the View Update Problem.



Asymmetric lens

Given an *update* from state $Y = GX$ in $\mathbf{Y}$ (with target Y') the asymmetric lens delivers (by a "putback" process P) an update to X in $\mathbf{X}$ (with target X') *along with compatible* re-synchronization data, that is $Y' = GX'$.

$$X \xmapsto{G} Y$$

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$$\begin{array}{ccc} X & \stackrel{G}{\longmapsto} & Y \\ & & \downarrow{\scriptstyle \beta} \\ & & Y' \end{array}$$

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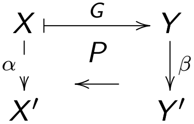

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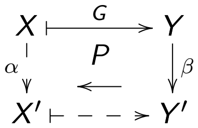


Asymmetric d-lens

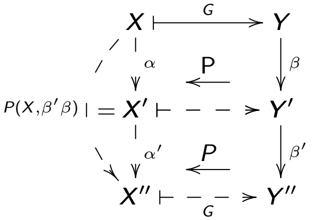
The formal axioms are:

An asymmetric d-lens is $L = (G, P)$
where $G : \mathbf{X} \longrightarrow \mathbf{Y}$ is the "Get" functor and
P is the "Put(back)" function and the data G, P satisfy:

(i) PutGet: $GP(X, \beta) = \beta$

(ii) PutId: $P(X, 1_{GX}) = 1_X$

(iii) PutPut:



$\alpha = P(X, \beta)$

$\alpha' = P(X', \beta')$

or

$$P(X, \beta'\beta : GX \longrightarrow Y' \longrightarrow Y'') = P(X', \beta' : GX' \longrightarrow Y'')P(X, \beta : GX \longrightarrow Y')$$

Asymmetric d-lens: examples

- Given a split op-fibration $G : \mathbf{X} \longrightarrow \mathbf{Y}$:
  Just define $P(X, \beta)$ to be the op-Cartesian arrow.

- For example, $d_0 : \mathbf{set^2} \longrightarrow \mathbf{set}$ or $d_1 : \mathbf{set^2} \longrightarrow \mathbf{set}$

- Or $V : \mathbf{C} \longrightarrow \mathbf{D}$ a small fully-faithful functor,
  $V^* : \widehat{\mathbf{D}} \longrightarrow \widehat{\mathbf{C}}$ is an opfibration

- This class called "c-lenses" by J & R and studied earlier (in the context of View Update Problem)
  - defined by equations analogous to asymmetric **set**-lens
  - algebras for a monad on $\mathbf{cat}/\mathbf{Y}$
  - the Put satisfies a "least change" property (to come)

- Indeed, an asymmetric *d-lens* is an algebra for a related *semi*-monad on $\mathbf{cat}/\mathbf{Y}$

- Note: *not every* asymmetric d-lens is an op-fibration

Symmetric lens and asymmetric d-lens

- Symmetric lenses compose, so do the asymmetric
  for set-based, category-based and other variants.
  NB: Lenses are the *morphisms.*

- *Span* of asymmetric d-lenses determines a symmetric lens:
  roughly: the **f** is the left leg Put, then the right leg Get

- Symmetric lens determines a span of asymmetrics
  roughly: head of span has squares w top/bottom *R*s

- Both have composition-compatible behaviour equivalence
  relations, that define suitable categories for an...

Symmetric lens and asymmetric d-lens

- Equivalence of categories from symmetric lenses to spans of asymmetrics
- The (asymmetric) c-lens special case has universality: the lifted updates are "least change" (to come)
- Question: what should "least change" mean for an arbitrary *symmetric* lens?
- Suggestion: a span of asymmetric, least change (c-)lenses?? *But not likely:* the head of the span is under-specified.

Construction

Motivated by database interoperation/integration
as implemented along a *common view*, we construct:

A *symmetric lens* L from a *cospan* of asymmetric d-lenses:

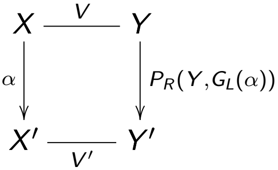
$$\mathbf{X} \xrightarrow{(G_L, P_L)} \mathbf{V} \xleftarrow{(G_R, P_R)} \mathbf{Y}$$

Set $L = (\delta_{\mathbf{X}}, \delta_{\mathbf{Y}}, \mathbf{f}, \mathbf{b}) : \mathbf{X} \longrightarrow \mathbf{Y}$ where

- $\mathbf{R}_{\mathbf{XY}} = \{(X, Y) \mid G_L X = G_R Y = V\}$
- $\delta_{\mathbf{X}}$ and $\delta_{\mathbf{Y}}$ projections from $\mathbf{R}_{\mathbf{XY}}$ to $\mid \mathbf{X} \mid$ and $\mid \mathbf{Y} \mid$, *and*

Construction...

- $\mathbf{f}(\alpha, (X, Y)) = (P_R(Y, G_L(\alpha)), (X', Y'))$ as in



  $Y' := d_1 P_R(Y, G_L(\alpha))$ and as $G_R(P_R(Y, G_L(\alpha))) = G_L(\alpha)$ we denote $V' = d_1 G_L(\alpha)$.

- Definition of **b** is similar.

Construction: Example

Recall that $d_1, d_0 : \mathbf{set^2} \longrightarrow \mathbf{set}$ are both Gets for asymmetric d-lens (even c-lens), with Puts from op-cartesian arrows.
Consider the construction for the cospan:

$$d_1 : \mathbf{set^2} \longrightarrow \mathbf{set} \longleftarrow \mathbf{set^2} : d_0$$

Let X, Y be objects of $\mathbf{set^2}$ with a synchronization
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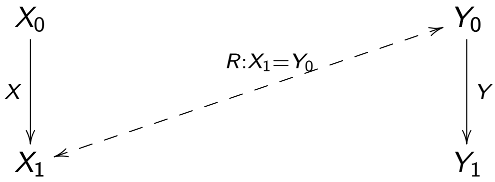

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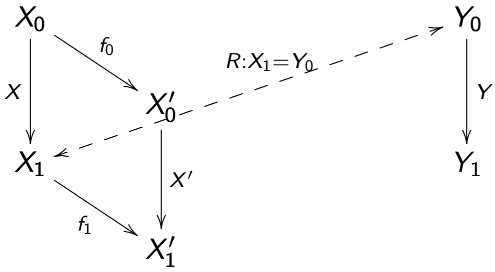
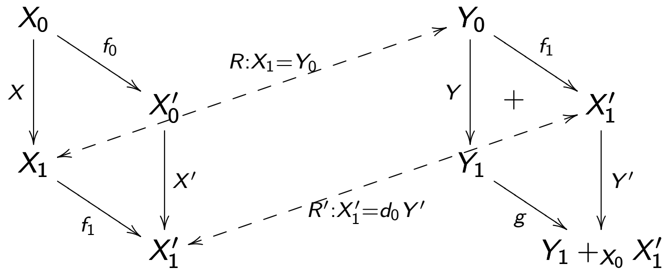
Let X, Y be objects of $\mathbf{set^2}$ with a synchronization
$R : X_1 = d_1 X = d_0 Y = Y_0$,
and if $(f_0, f_1) : X \longrightarrow X'$ a left side update, as in



Construction: Example

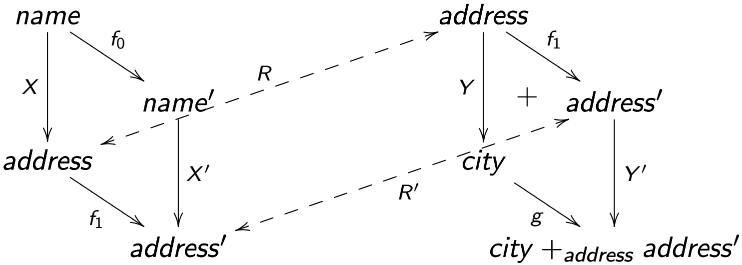
Forward propagation constructs a new arrow $(f_1, g) : Y \longrightarrow Y'$ using the pushout, together with a new synch $R' : X_1' = d_0 Y'$:



Construction: Example

For example: a left hand db state assigns *name* to *address;*
a right hand state assigns *address* to *city;*
so a synchronization is an *address* matching

name/address update propagates to a right hand update,
also creating a new *city* set: the pushout



Compatibility 1

G_L and G_R in the construction define a cospan of object functions, so *at most one synchronization* for a pair X and Y: the object V of $\mathbf{V}$ both X and Y map to.

The cospan similarly defines a relation from *arrows* of $\mathbf{X}$ to *arrows* of $\mathbf{Y}$, compatible with synchronization:

Definition

In a cospan of asymmetric d-lenses

$$\mathbf{X} \xrightarrow{(G_L,P_L)} \mathbf{V} \xleftarrow{(G_R,P_R)} \mathbf{Y}$$

arrows α of $\mathbf{X}$ and β of $\mathbf{Y}$ are called compatible if $G_L(\alpha) = G_R(\beta)$.

Universality

In case (G_L, P_L) and (G_R, P_R) are *c-lenses*, we find a *universal property* for the **f** and **b** constructed above.

First, notation: to $\alpha : X \longrightarrow X'$ and synchronization $(X, Y) = V$:
Write $\mathbf{f}(\alpha, (X, Y)) = (\mathbf{f}(\alpha, V)_0, \mathbf{f}(\alpha, V)_1)$
with two components, as in the square.

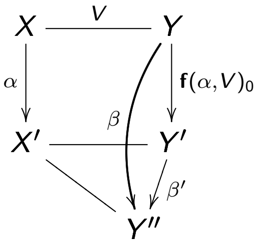
$$\begin{array}{ccc} X & \overset{V}{\text{———}} & Y \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \mathbf{f}(\alpha,V)_0} \\ X' & \underset{\mathbf{f}(\alpha,V)_1}{\text{———}} & Y' \end{array}$$

Universality

The universal property satisfied by $\mathbf{f}(\alpha, (X, Y))$:

Proposition

Given a cospan of c-lenses $\mathbf{X} \xrightarrow{(G_L, P_L)} \mathbf{V} \xleftarrow{(G_R, P_R)} \mathbf{Y}$ *and* $\alpha : X \longrightarrow X'$ *and* $V = (X, Y)$. *Then, for any* $\beta : Y \longrightarrow Y''$ *compatible with* α *there is a unique* $\beta' : Y' \longrightarrow Y''$ *satisfying* $\beta = \beta' \mathbf{f}(\alpha, V)_0$ *and* $G_R(\beta') = 1_{G_L X'}$.



Remarks

- Thus among arrows with domain Y *compatible* with α, $\mathbf{f}(\alpha, (X, Y))_0$ is the least-change arrow
- That is, other compatible updates factor through β via *arrow compatible with the identity* on X'
- Similarly for **b**
- *Spans* of c-lenses determine relations on objects of **X** and **Y** but with no obvious universal property
- And what about more general symmetric lenses...?

Compatibility 2

- When is a symmetric lens "least change"?
- We'll require not just a synchronization of objects but also a compatibility for arrows of **X** and the arrows of **Y**.
- In forward (or backward) propagation square, the left & right arrows surely must be compatible, so

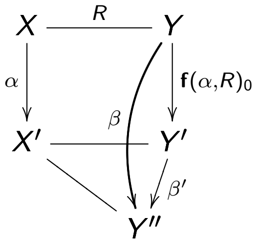
Definition

Let $L = (\delta_{\mathbf{X}}, \delta_{\mathbf{Y}}, \mathbf{f}, \mathbf{b})$ be a symmetric lens from **X** to **Y** with $\mathbf{R_{XY}}$.
A compatibility relation C on L from arrows **X** to arrows of **Y**:
– contains the union of pairs from **f**-squares and **b**-squares, and
– if $\alpha \, C \, \beta$, there are R_0, R_1 in $\mathbf{R_{XY}}$ with $R_0 : d_0(\alpha) \leftrightarrow d_0(\beta)$, and for d_1

Least change lens

Definition

A symmetric lens L equipped with a compatibility relation C is called least-change if for $\alpha : X \longrightarrow X'$ and $R : X \leftrightarrow Y$ we have $\mathbf{f}(\alpha, R)$ satisfies the following:
For $\beta : Y \longrightarrow Y''$ compatible w α there is unique $\beta' : Y' \longrightarrow Y''$ with $\beta = \beta' \mathbf{f}(\alpha, R)_0$ and $1_{X'} \, C \, \beta'$:



And similarly for back propagation **b**.

Remarks

- So least-change symmetric lens with compatibility relation has **f** and **b** satisfying the universal property from cospans of c-lenses:
- Least-change arises from conditions a symmetric lens *with compatibility* may satisfy, *not* just a property of a cospan of c-lenses
- Question: What least-change symmetric lenses have compatibility from a cospan of asymmetric d-lenses, and when from a cospan of c-lenses?
- Question: Symmetric lens always corresponds to a span of asymmetric d-lenses; when does a least change compatibility relation arise from the span?

A symmetric lens with compatibility relation is represented by a span or cospan of asymmetric d-lenses if forwards and backwards propagations have the same effects *and* the synchronization, and the compatibility relations are the same.

Complete compatibility

Not every symmetric lens can be represented by a cospan of asymmetric d-lenses:

Relation R from A to B, is called complete if it has a complete bipartite graph.

Proposition

The compatibility relation from a cospan of asymmetric d-lenses is a coproduct of complete relations.

But, this strong condition is not always satisfied.

Conclusion

- Symmetric lenses from c-lens *spans* do not char'ze universality
- Proposal above for *least change* symmetric lens
- Symmetrics from c-lens cospans *are* least change
- We have necessary conditions for least change symmetric lenses, but...
- Characterization question remains open.
- Some urls:
- `www.mta.ca/~rrosebru`
- `www.comp.mq.edu.au/~mike/`

Thanks!