Fundamental Groupoids for Orbifolds

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joint with Dorette Pronk and Courtney Thatcher

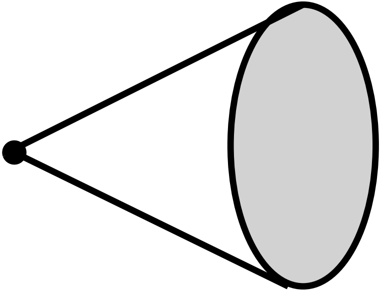
Orbifolds

An orbifold is:

- a generalization of a manifold
- a space that is locally modelled by quotients of $\mathbb{R}^n$ by actions of finite groups
- allows controlled singularities

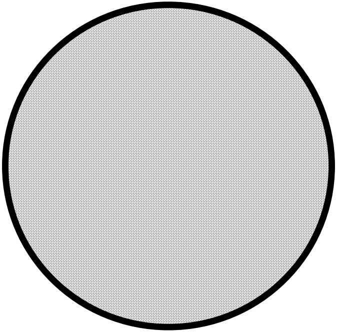
Example 0: A Manifold (with boundary)

Example 1: A Cone Point

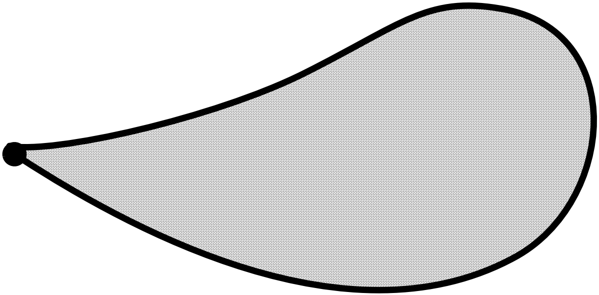


Example 2: Silvered Interval

Example 3: Mirrored Boundary Disk

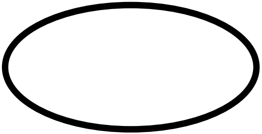


Example 4: The Teardrop



Example 5: The Billiard Table

Example 6: Ineffective Z/3 Action



Z/3 Isotropy

Orbifolds via Atlases (Effective Edition)

We can represent an orbifold using charts making up an atlas:

- $\widetilde{U}$ is a connected open subset of $\mathbb{R}^n$;
- G is a finite group acting *effectively* on $\widetilde{U}$;
- $\pi : \widetilde{U} \to U$ is a continuous and surjective map that induces a homeomorphism between U and $\widetilde{U}/G$

Orbifolds via Atlases (Effective Edition)

Charts creating an atlas:

- A collection of charts $\mathcal{U}$ such that the quotients cover the underlying space, and all chart embeddings between them.
- The charts are required to be locally compatible: for any two charts for subsets $U, V \subseteq X$ and any point $x \in U \cap V$, there is a neighbourhood $W \subseteq U \cap V$ containing x with a chart $(\widetilde{W}, G_W, \pi_W)$ in $\mathcal{U}$, and chart embeddings into $(\widetilde{U}, G_U, \pi_U)$ and $(\widetilde{V}, G_V, \pi_V)$.

Orbifolds via Atlases (Effective Edition)

For any μ_{ji} in $\mathcal{O}(\mathcal{U})$, the set *Emb*(μ_{ji}) forms an atlas bimodule

$$\mathrm{Emb}(\mu_{ji}) : G_i \longrightarrow\!\!\!\!\!\!\!\!\;\mapsto G_j.$$

with actions given by composition. If $i = j$ the atlas bimodule Emb(μ_{ii}) is isomorphic to the trivial bimodule G$_i$ associated to the group G_i. Furthermore, these define a pseudofunctor

$$\mathrm{Emb} : \mathcal{O}(\mathcal{U}) \longrightarrow \mathrm{GroupMod},$$

with Emb$(U_i) := G_i$ on objects, and Emb$(\mu_{ji}) : G_i \longrightarrow\!\!\!\!\!\!\!\!\;\mapsto G_j$ on morphisms.

Orbifolds via Atlases (General Edition)

Let U be a non-empty connected topological space; an *orbifold chart* (also known as a *uniformizing system*) of dimension n for U is a quadruple $(\widetilde{U}, G, \rho, \pi)$ where:

- $\widetilde{U}$ is a connected and simply connected open subset of $\mathbb{R}^n$;
- G is a finite group;
- $\rho : G \to \mathrm{Aut}(\widetilde{U})$ is a (not necessarily faithful) representation of G as a group of smooth automorphisms of $\widetilde{U}$; we set $G^{\mathrm{red}} := \rho(G) \subseteq \mathrm{Aut}(\widetilde{U})$ and $\mathrm{Ker}(G) := \mathrm{Ker}(\rho) \subseteq G$;
- $\pi : \widetilde{U} \to U$ is a continuous and surjective map that induces a homeomorphism between U and $\widetilde{U}/G^{\mathrm{red}}$.

Orbifolds via Atlases (General Edition)

An *orbifold atlas* of dimension n for X is:

1. a collection $\mathcal{U} = \{(\widetilde{U}_i, G_i, \rho_i, \pi_i)\}_{i \in I}$ of orbifold charts, of dimension n, connected and simply connected, such that the reduced charts $\{(\widetilde{U}_i, G_i^{\mathrm{red}}, \pi_i)\}_{i \in I}$ form a Satake atlas for X; let $(\mathrm{Con}, \gamma) \colon \mathcal{O}(\mathcal{U}) \to \mathrm{GroupMod}$ be the induced pseudofunctor
2. a pseudofunctor

$$\mathrm{Abst} : \mathcal{O}(\mathcal{U}) \longrightarrow \mathrm{GroupMod}$$

such that for each $i \in I$, $\mathrm{Abst}(U_i) = G_i$ and for each μ_{ji} in $\mathcal{O}(\mathcal{U})$, $\mathrm{Abst}(\mu_{ji})$ is an atlas bimodule $G_i \longrightarrow\!\!\!\!\!\!\!+ \;\, G_j$, (i.e., the left action of G_j is free and transitive and the right action of G_i is free).

Orbifolds via Atlases (General Edition)

(3) an oplax transformation $\boldsymbol{\rho} = (\{\boldsymbol{\rho}_i\}_{i\in I}, \{\rho_{ji}\}_{i,j\in I, U_i\subseteq U_j})$: Abst $\Rightarrow$ Con: each ρ_i is a group homomorphism from G_i to G_i^{red}, hence it induces a bimodule $\boldsymbol{\rho}_i : G_i \relbar\joinrel\nrightarrow G_i^{\mathrm{red}}$ forming the components of the transformation. We further require that:

- the ρ_{ji} are surjective maps of bimodules;
- (transitivity on the kernel) whenever $\rho_{ji}(e_j^{\mathrm{red}} \otimes \lambda) = \rho_{ji}(e_j^{\mathrm{red}} \otimes \lambda')$ for $\lambda, \lambda' \in \mathrm{Abst}(\mu_{ji})$, there is an element $g \in G_i$ such that $\lambda \cdot g = \lambda'$ (here e_j^{red} is the identity element of G_j^{red}).

Orbifolds via *G*-spaces

We can represent some (most?) orbifolds via group actions

- the orbifold is the quotient space of a (compact Lie) group acting on a manifold
- if the group is finite, the orbifold is a global quotient
- unknown whether all orbifolds are representable this way

Orbifolds via Topological Groupoids

- A **topological groupoid** has a space of object $\mathcal{G}_0$ and a space of arrows $\mathcal{G}_1$, where all structure maps are continuous
- $\mathcal{G}$ is **étale** when s (and hence t) is a local homeomorphism
- $\mathcal{G}$ is **proper** when the diagonal,

  $$(s,t)\colon \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0,$$

  is a proper map (i.e., closed with compact fibers).

Orbigroupoids

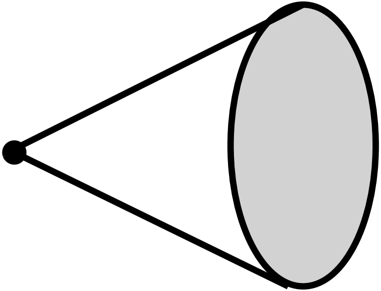
Definition

- A topological groupoid is an **orbigroupoid** if it is both étale and proper.
- All isotropy groups are finite.
- The quotient space,

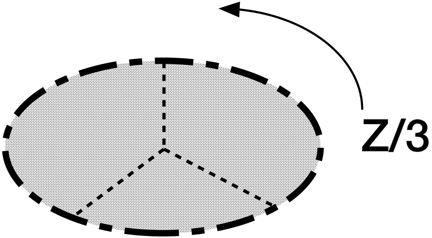
$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathcal{G}_0 \arrow[r, two heads] & X_{\mathcal{G}} \end{tikzcd}$$

  is also called the **underlying space** of the orbigroupoid.
- This space is an orbifold.

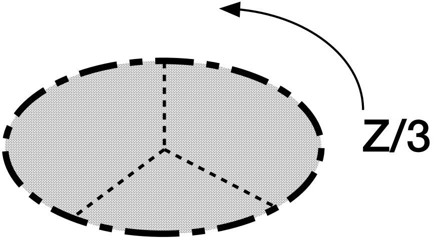
Example 1: A Cone Point



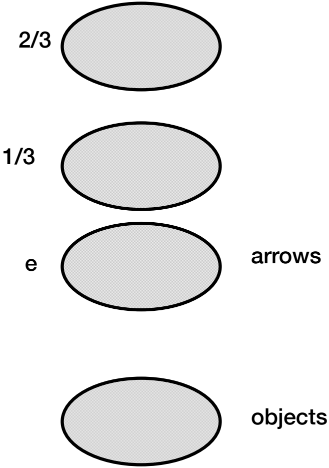
Example 1: A Cone Point as an atlas (with one chart)



Example 1: A Cone Point as a *G*-space

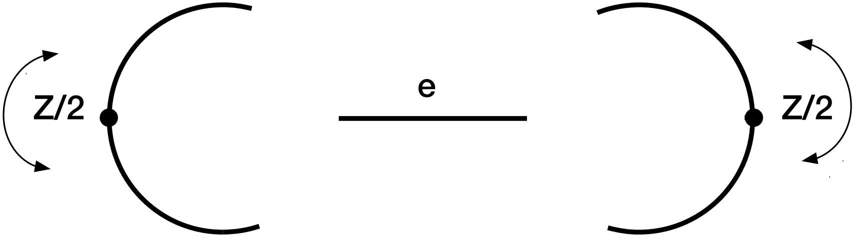


Example 1: A Cone Point as a groupoid



Example 2: Silvered Interval

Example 2: Silvered Interval as an atlas



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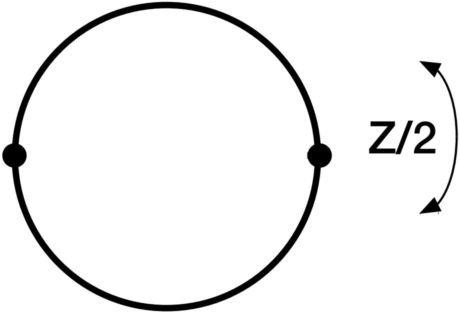

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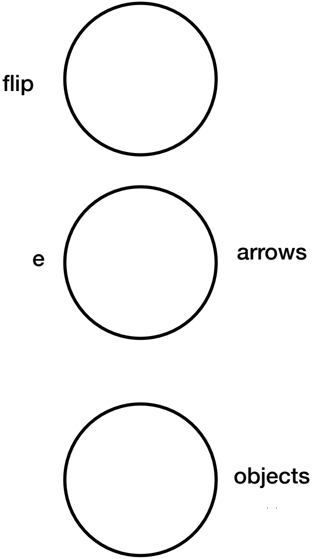
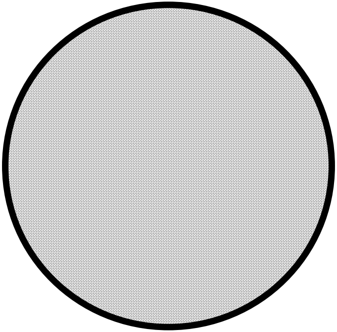


Example 3: Mirrored Boundary Disk



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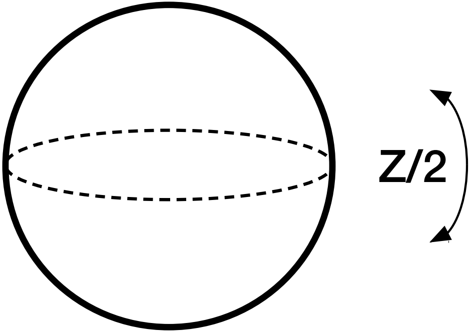

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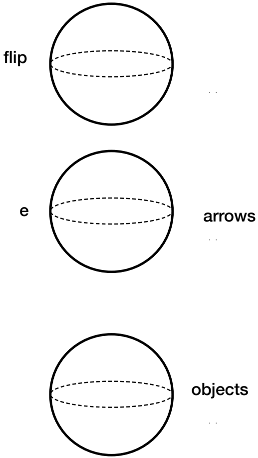
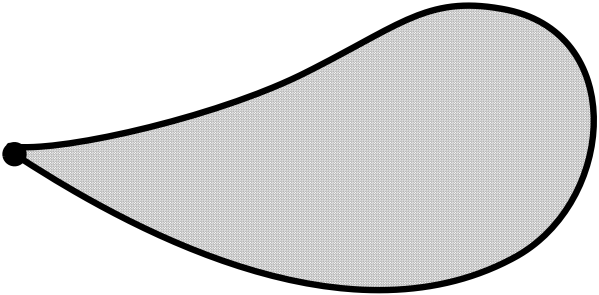


Example 4: The Teardrop



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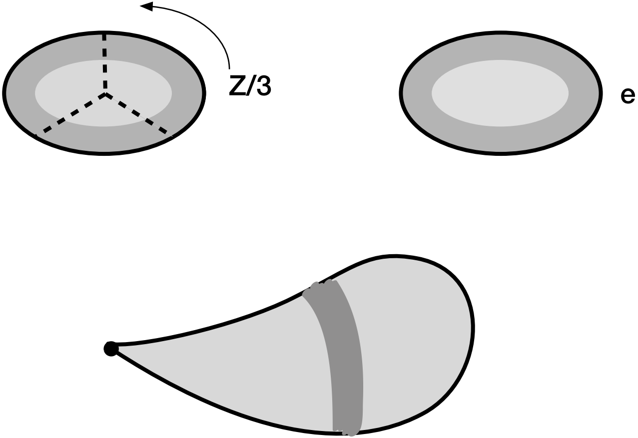

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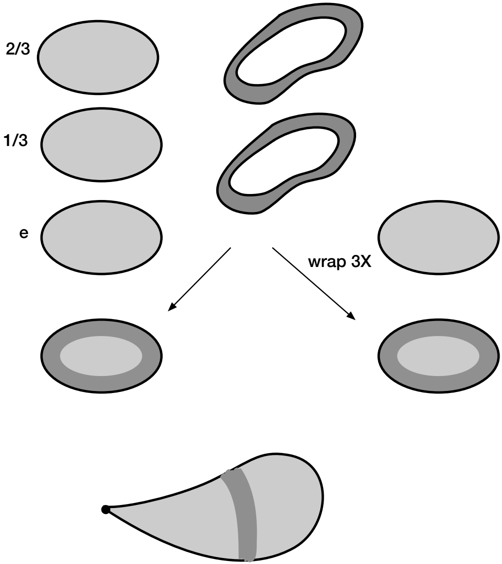

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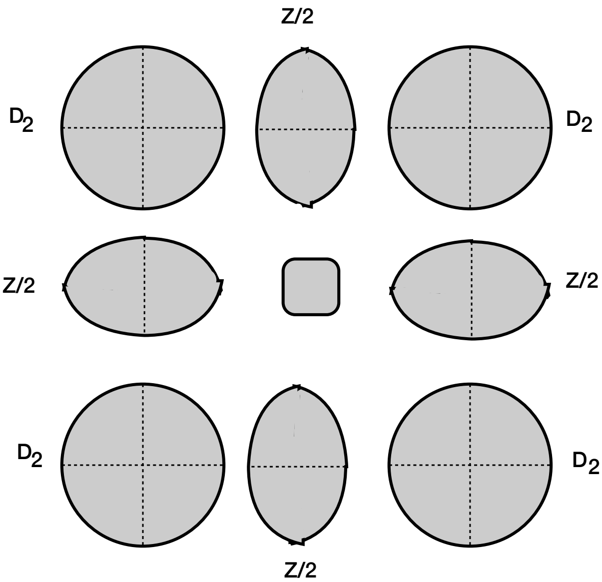
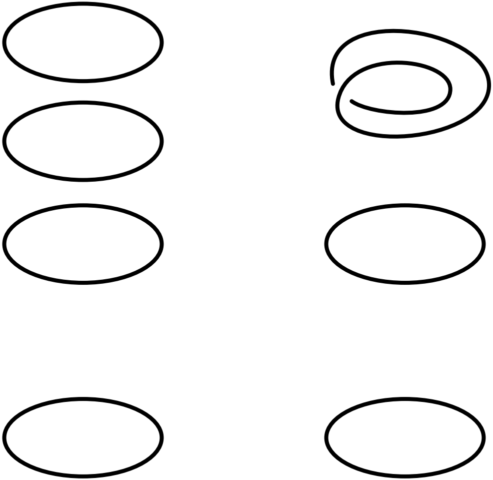


Example 6: Ineffective Z/3 Action



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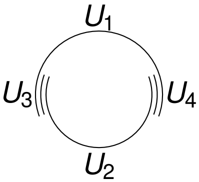
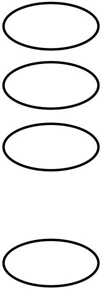


$G_{U_i} = \mathbb{Z}/3$ and $G^{\mathrm{red}}_{U_i} = \{e\}$. Forr each inclusion $\mu_{ji} : U_i \hookrightarrow U_j$, we need a module M_{ji} and a map of bimodules ρ_{ji} as follows:

$$\begin{tikzcd}
\mathbb{Z}/3 = \{e, \omega_i, \omega_i^2\} \arrow[r, "\mathrm{Abst}(\mu_{ji})=\mathrm{M}_{ji}", "/" marking] \arrow[d, "\boldsymbol{\rho}_i"', "/" marking] \arrow[dr, phantom, "\Swarrow\ \rho_{ji}"] & \mathbb{Z}/3 = \{e, \omega_j, \omega_j^2\} \arrow[d, "\boldsymbol{\rho}_j", "/" marking] \\
\{e\} \arrow[r, "\mathrm{Con}(\mu_{ji})=\{\lambda_{ji}\}"', "/" marking] & \{e\}
\end{tikzcd} \tag{1}$$

Example 6: Ineffective Z/3 Action



left multiply by ω_j

$$\begin{tikzcd}
a_{ji} \arrow[r] & b_{ji} \arrow[r] & c_{ji} \arrow[ll, bend right]
\end{tikzcd}$$

right multiply by ω_i

$$\begin{tikzcd}
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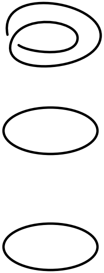
Example 6: Ineffective Z/3 Action



M_{13}, M_{14} and M_{23} as before, M_{24} with action given by

left multiply by ω_j

$$\begin{tikzcd} a_{ji} \arrow[r] & b_{ji} \arrow[r] & c_{ji} \arrow[ll, bend right] \end{tikzcd}$$

right multiply by ω_i

$$\begin{tikzcd} a_{ji} \arrow[rr, bend left] & b_{ji} \arrow[l] & c_{ji} \arrow[l] \end{tikzcd}$$

(Borel) Fundamental Group

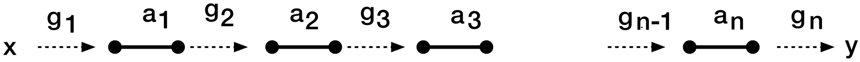
If $\mathcal{G}$ is a groupoid representing an orbifold, we can define a fundamental group by:

- $\pi_1(B\mathcal{G})$
- Haefliger paths
- deck transformation of universal cover
- homotopy classes of maps $I \to \mathcal{G}$

Haefliger paths

Let $\mathcal{G}$ be a Lie groupoid. A path from x to y in $\mathcal{G}_0$ is:

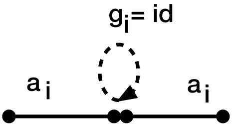
- a subdivision $0 = t_0 < t_1 < t_2 \ldots t_n = 1$
- a sequence $(g_0, \alpha_1, g_1, \ldots, \alpha_n, g_n)$
- $g_i \in \mathcal{G}_1$ such that $s(g_0) = x, t(g_n) = y$
- $\alpha_i : [t_{i-1}, t_i] \to \mathcal{G}_0$ is a path from $t(g_{i-1})$ to $s(g_i)$



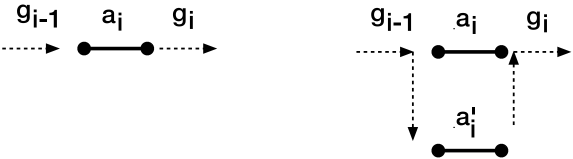
Haefliger Paths

Two paths are equivalent if:

- we add a new point to the subdivision with an identitiy g_i:



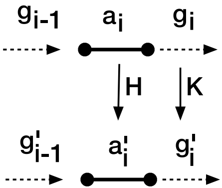
- we have homotopy $h : [t_{i-1}, t_i] \to \mathcal{G}_1$ with $s \circ h_i = \alpha_i$ and $t \circ h_i = \alpha'_i$ and we replace $(\dots g_{i-1}, \alpha_i, g_i, \dots)$ by $(\dots h(t_{i-1}) g_{i-1}, \alpha', g_i h(t_i)^{-1}, \dots)$



Haefliger Paths

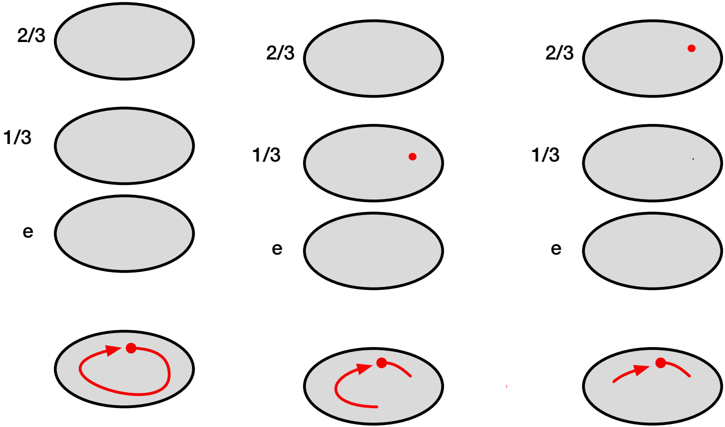
Two paths are homotopic if:

- we have homotopies $h : [t_{i-1}, t_i] \times I \to \mathcal{G}_0$ with $h(t, 0) = \alpha_i$ and $h(t, 1) = \alpha'_i$
- we have compatible homotopies $K : I \to \mathcal{G}_1$ with $K(0) = g_i$ and $K(1) = g'_i$



We define the orbifold fundamental groupoid as the homotopy classes of these paths.

Order 3 Cone



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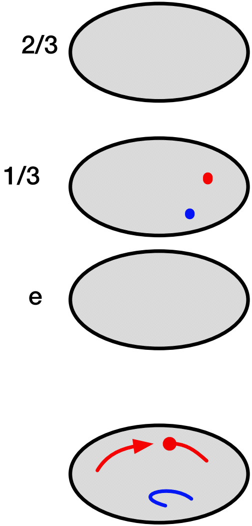

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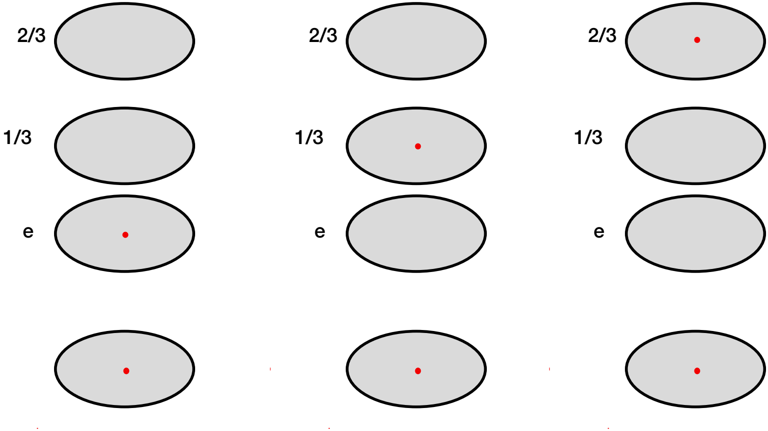

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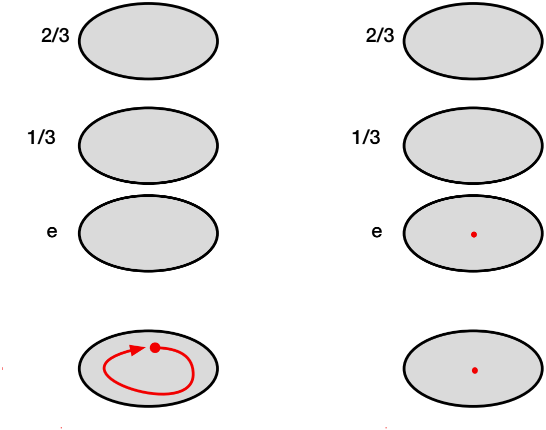

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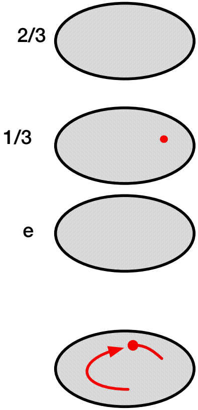

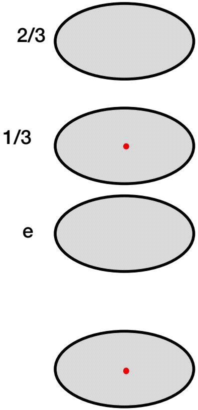

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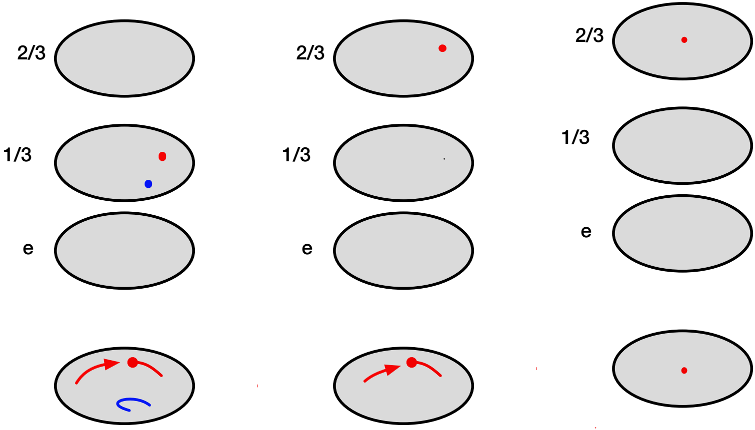

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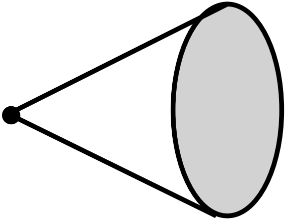
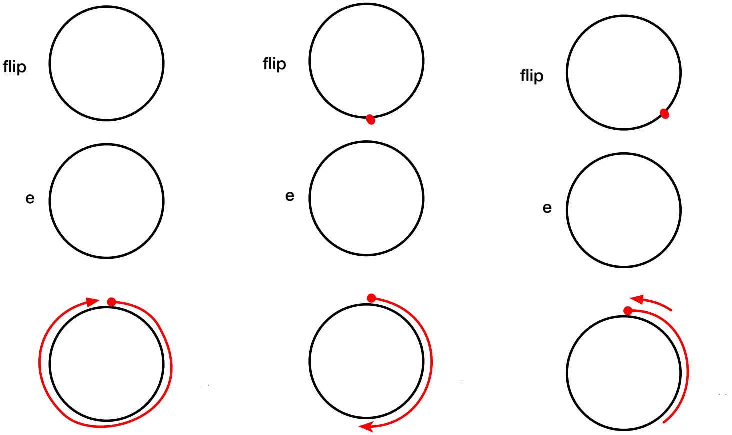


$\pi_1(\mathcal{G}) = \mathbb{Z}/3$

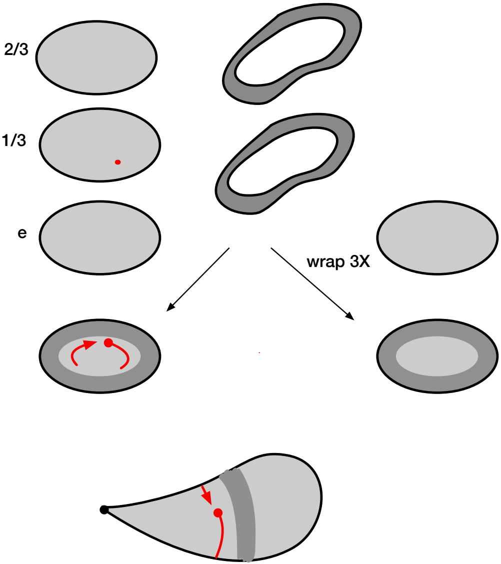
Silvered Interval



Silvered Interval

$$\pi_1(\mathcal{G}) = D_\infty$$

Teardrop



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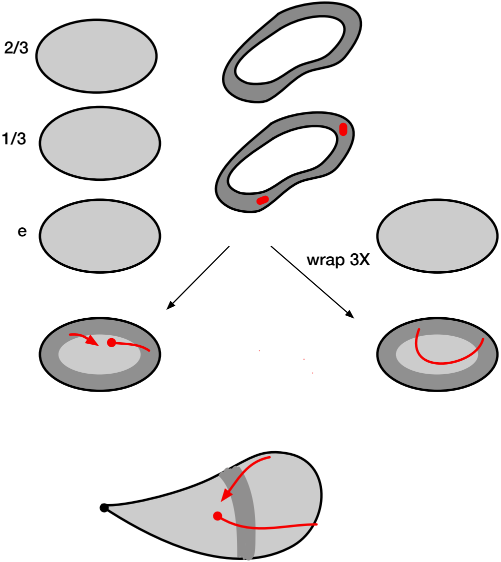

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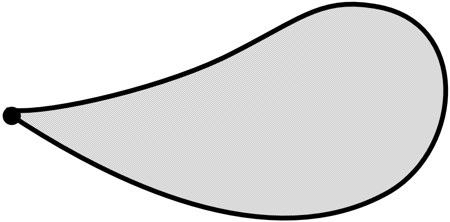


$\pi_1(\mathcal{G}) = e$

(Borel) Fundamental Group

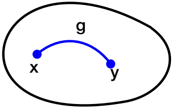
Recall we can define $\pi_1(\mathcal{G})$ by:

- $\pi_1(B\mathcal{G})$
- Haefliger paths
- deck transformation of universal cover
- homotopy classes of maps $I \to \mathcal{G}$

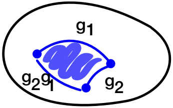
(Borel) Fundamental Group

$B\mathcal{G}$ defined by the geometric realization of the nerve of $\mathcal{G}$:

- Δ^0 for every $x \in \mathcal{G}_0$
- Δ^1 for every $g \in \mathcal{G}_1$ attached to $s(g)$ and $t(g)$



- Δ^2 for every composible (g_1, g_2) attached by $g_1, g_2, g_2 g_1$



- higher simplices attached but do not affect π_1

(Borel) Fundamental Group

$\pi_1(B\mathcal{G})$ is the Haefliger group

- a path in $\pi_1(BG)$ can follow a line in $B\mathcal{G}$ corresponding to $g \in \mathcal{G}_1$, giving a hop
- paths can be homotopic over triangles corresponding to equivalence of Haefliger paths

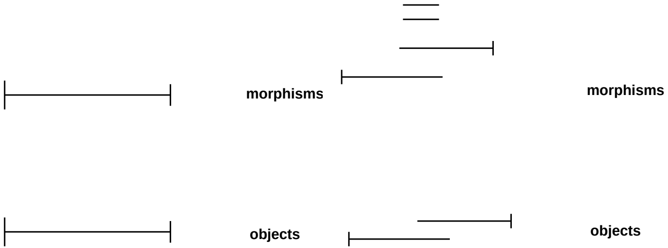
(Borel) Fundamental Group

Defined via deck transformations (topos)

Defined via homotopy classes of maps $I \to \mathcal{G}$:

Morita Equivalence

- The following two groupoids both represent the unit interval as orbispace



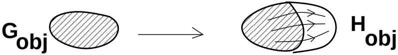
- They are not isomorphic in the category of orbigroupoids and groupoid homomorphisms.
- However, the groupoid homomorphism from the second to the first is an essential equivalence.

Essential Equivalences

- A morphism $f\colon \mathcal{G} \to \mathcal{H}$ is an **essential equivalence** when it is essentially surjective and fully faithful.
- It is **essentially surjective** when $\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \longrightarrow \mathcal{H}_0$ in

$$\begin{tikzcd}
\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \arrow[r] \arrow[d] & \mathcal{H}_1 \arrow[r, "t"] \arrow[d, "s"] & \mathcal{H}_0 \\
\mathcal{G}_0 \arrow[r, "f_0"'] & \mathcal{H}_0 &
\end{tikzcd}$$

is an **open surjection**.



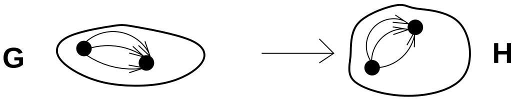
f may not be onto the objects of $\mathcal{H}$, but every object in $\mathcal{H}_0$ is isomorphic to an object in the image of $\mathcal{G}_0$.

Essential Equivalences

The morphism $f : \mathcal{G} \to \mathcal{H}$ is **fully faithful** when

$$\begin{tikzcd}
G_1 \arrow[r, "\phi"] \arrow[d, "{(s,t)}"'] & H_1 \arrow[d, "{(s,t)}"] \\
G_0 \times G_0 \arrow[r, "\phi\times\phi"] & H_0 \times H_0
\end{tikzcd}$$

is a **pullback**,



The local isotropy structure is preserved.

Morita Equivalence

- The equivalence relation generated by the essential equivalences is called **Morita Equivalence**
- Orbigroupoids represent the same orbispace if and only if they are Morita equivalent
- To define a category of orbispaces, we use a **bicategory of fractions** to invert the essential equivalences

Generalized Maps

- Maps are **generalized maps** defined by spans

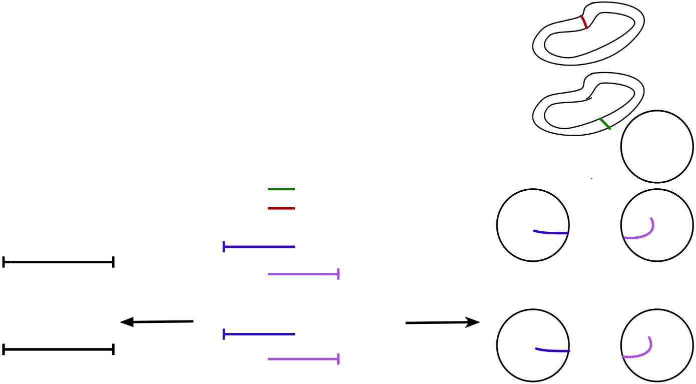
$$\mathcal{G} \xleftarrow{\upsilon} \mathcal{K} \xrightarrow{\varphi} \mathcal{H}$$

  where υ is an essential equivalence
- A **2-cell** between two generalized maps is an (equivalence class of) diagrams

$$\begin{tikzcd}
 & & \mathcal{K} \arrow[dll, "\upsilon"'] \arrow[drr, "\varphi"] & & \\
\mathcal{G} & \alpha_1\Downarrow & \mathcal{L} \arrow[u, "\nu_1"'] \arrow[d, "\nu_2"] & \alpha_2\Downarrow & \mathcal{H} \\
 & & \mathcal{K}' \arrow[ull, "\upsilon'"] \arrow[urr, "\varphi'"'] & &
\end{tikzcd}$$

  where $\upsilon\nu_1$ is an essential equivalence.

Example



Equivariant Homotopy Perspective

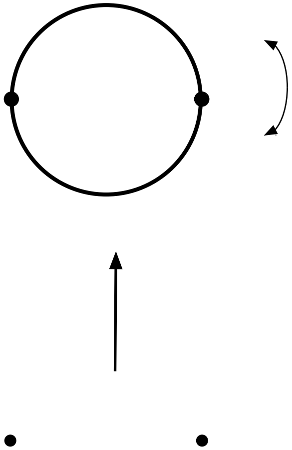
- Fix a group G, let X be a G-space.
- A 'point' $x \in X$ comes with a whole orbit $\{gx \,|\, g \in G\}$
- Define the fixed set $X^H = \{x \in X \mid hx = x \ \forall \ h \in H\}$
- A G-map $x : G/H \to X$ is equivalent to a point in X^H: $x \longleftrightarrow x(eH)$.
- we think of G-spaces as diagrams of fixed sets
- organized by $\mathcal{O}_G$: category with
  - objects G/H
  - morphisms G-maps

Example: $\mathbb{Z}/2$

- Example: $G = \mathbb{Z}/2$, $\mathcal{O}_G$ has two objects, G/G and G/e two non-identity maps:
  projection $G/e \to G/G$
  a non-trivial self-map $G/e \to G/e$.

$$\begin{tikzcd}
G/e \arrow[loop above, "\tau"] \arrow[d, "\rho"] & & \tau^2 = id \\
G/G & & \tau\rho = \rho
\end{tikzcd}$$

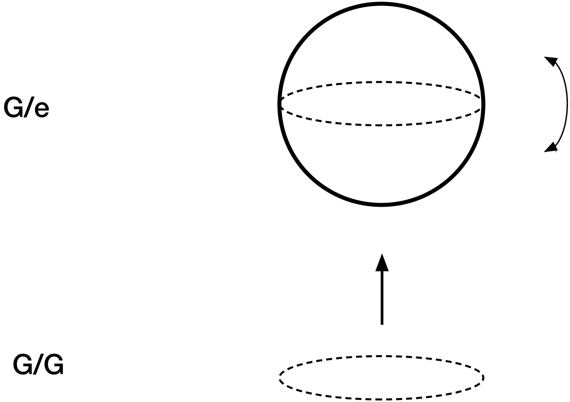
Silvered interval as $\mathbb{Z}/2$-space



G/e

G/G

Mirrored disk as $\mathbb{Z}/2$-space

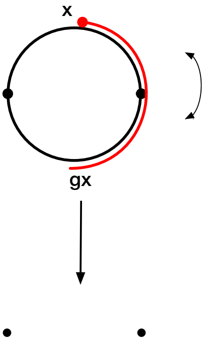


tom Dieck Fundamental Category

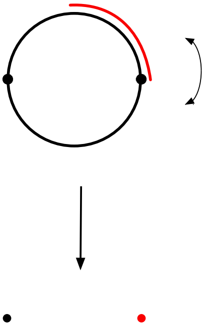
The equivariant fundamental category $\prod_G(X)$:

- look at the functor $\mathcal{O}_G \to \mathit{Gpds}$ defined by $\Pi(X^H)$
- define the Grothendieck colimit $\int_{\mathcal{O}_G} \Pi(X^-)$
- objects are given by $(G/H, x)$ where $x \in X^H$
- arrows: $(G/H, x)$ to $(G/K, y)$ is given by (α, γ) where $\alpha : G/H \to G/K$ in $\mathcal{O}_G$ and γ is a path with $\gamma_0 = x$ and $\gamma_1 = y\alpha$

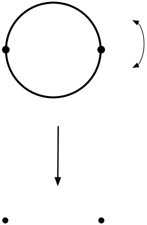
Silvered interval as $\mathbb{Z}/2$-space



Silvered interval as $\mathbb{Z}/2$-space

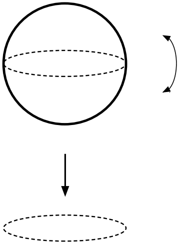


Silvered interval as $\mathbb{Z}/2$-space



$$\begin{tikzcd} & D_\infty \arrow[dl] \arrow[dr] & \\ e & & e \end{tikzcd}$$

Mirrored disk as $\mathbb{Z}/2$-space



$$\begin{tikzcd} e \arrow[d] \\ \mathbb{Z} \end{tikzcd}$$

tom Dieck fundamental group for orbifolds

- The Borel fundamental groupoid gives the tom Dieck $\prod_G$ at G/e
- We want a category that has all of it
- Challenges
  - Local structures can be for different groups - how to patch together to get a global $\mathcal{O}_G$ category?
  - Morita invariance

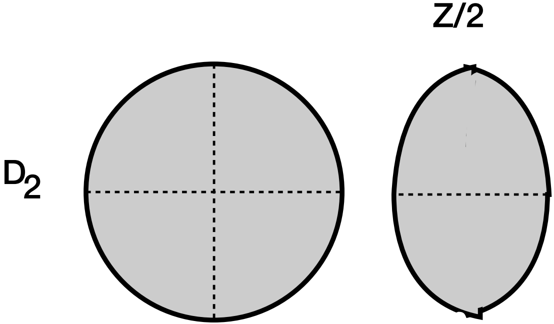
Idea: Representable Orbifolds

- many (maybe all?) orbifolds can be represented as quotients of compact Lie group actions
- we can define the tom Dieck $\prod_G$ for these
- it will not be Morita invariant
- however, a discrete version is

Idea: van Kampen

- Use a van Kampen to define a pushout of the local categories?
- Problem: we seem to be getting Cech information included

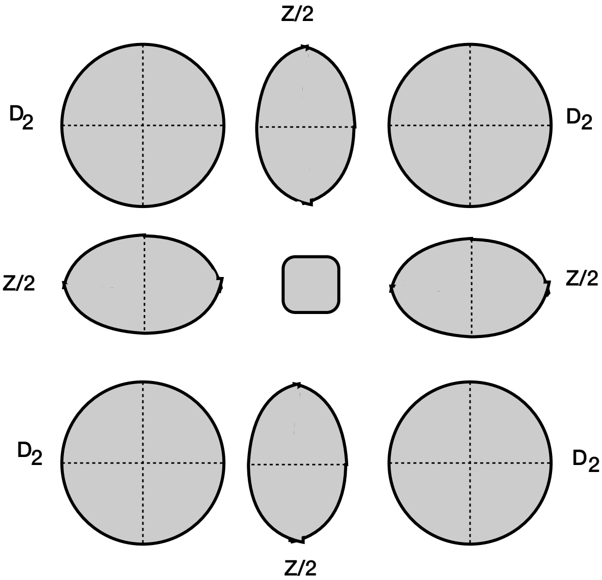
Example 5: The Billiard Table



Idea: van Kampen

$$\begin{tikzcd}
 & D_2 \arrow[dl] \arrow[d] \arrow[dr] & & \\
D_2/\tau \arrow[dr] & D_2/\sigma\tau \arrow[d] & D_2/\sigma^2\tau \arrow[dl] \arrow[r, dashed] & \mathbb{Z}/2/e \arrow[loop above, "\tau"] \arrow[d, "\rho"] \\
 & D_2/D_3 & & \mathbb{Z}/2/\mathbb{Z}/2
\end{tikzcd}$$

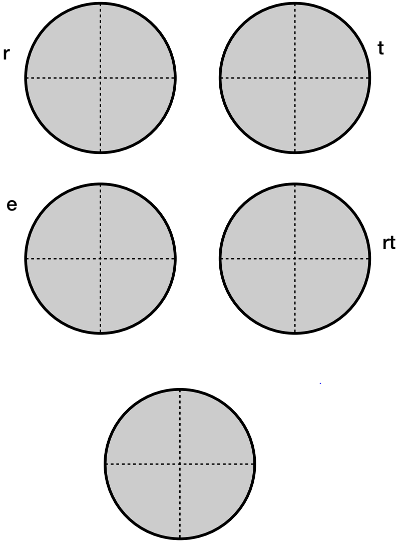
Our category wraps around



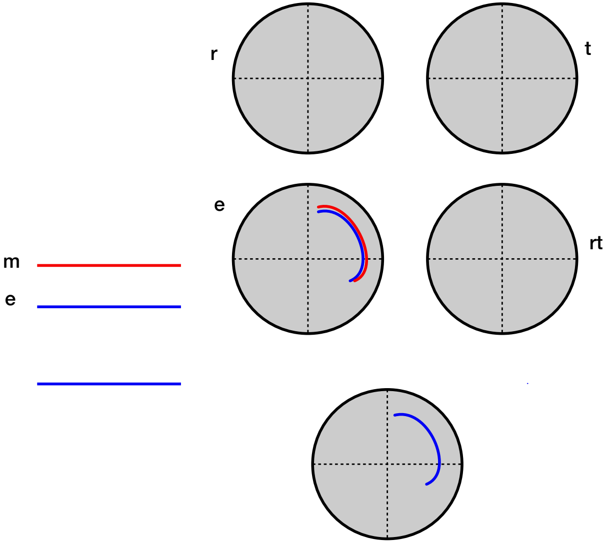
Idea: Use generalized maps

- the Borel group is given by generalized maps $[I, \mathcal{G}]$
- try defining $[I_K, \mathcal{G}]$
- this seems to get the fixed point data, but not the connections between the strata?

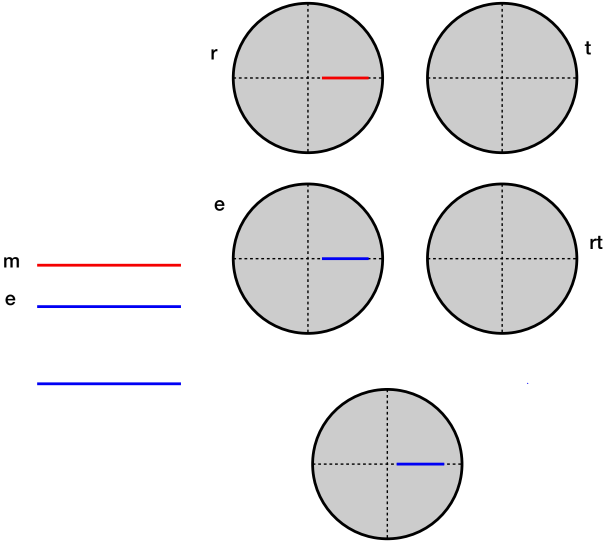
Example 5: The Billiard Table



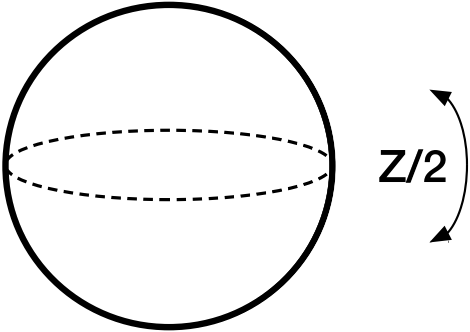
Example 5: The Billiard Table



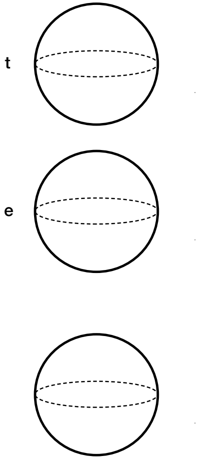
Example 5: The Billiard Table



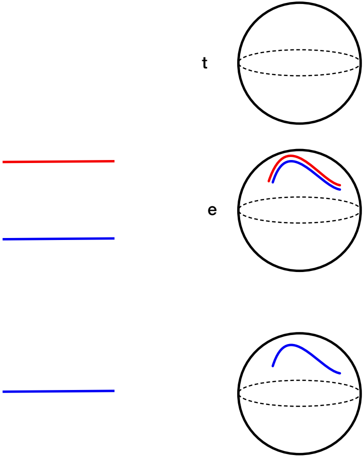
Example 3: Mirrored Boundary Disk



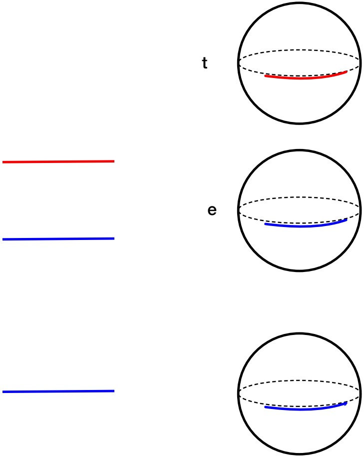
Example 3: Mirrored Boundary Disk



Example 3: Mirrored Boundary Disk



Example 3: Mirrored Boundary Disk



$$[I_{\mathbb{Z}/2}, \mathcal{G}] = e \coprod \mathbb{Z}$$

Sectors

- $\Lambda\mathcal{G}$ is the inertia groupoid $\Lambda\mathcal{G} = \{g \in \mathcal{G}_1 \mid s(g) = t(g)\}$
- $[I_{\mathbb{Z}}, \mathcal{G}] = \pi_1(\Lambda\mathcal{G})$
- $[I_{\mathbb{Z}\star\mathbb{Z}\star\mathbb{Z}\ldots}, \mathcal{G}] = \pi_1(\tilde{\Lambda}\mathcal{G})$ where $\tilde{\Lambda}\mathcal{G}$ is the multisectors
- $\tilde{\Lambda}\mathcal{G}$ has all the fixed sets
- but both of these produce disjoint components