Hopf categories as Hopf monads in enriched matrices

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Outline

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1. Hopf monads in monoidal bicategories

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3. Monoidal fibrant double context

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4. Further examples

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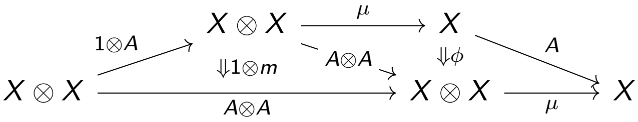
- A *Hopf monad* is a bimonad for which the *Hopf map* is iso:



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- Galois map invertibility (*well-pointed*): for A-comodule (N, ψ)

$$N \circ A \xrightarrow{\psi \circ 1} (A \bullet N) \circ (A \bullet J) \xrightarrow{\xi} (A \circ A) \bullet (N \circ J) \xrightarrow{m \bullet 1} A \bullet (N \circ J).$$

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$$X \underset{\mu}{\overset{\delta}{\rightleftarrows}} X \times X \quad \text{by } \delta_{x,y,z} = \mu_{y,z,x} = \begin{cases} I, & \text{if } x=y=z \\ 0, & \text{otherwise} \end{cases}$$

$$X \underset{\eta}{\overset{\epsilon}{\rightleftarrows}} 1 \quad \text{by } \epsilon_x = \eta_x = I, \qquad \delta \dashv \mu \text{ and } \epsilon \dashv \eta.$$

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Hopf monads in $\mathcal{V}$-**Mat** are Hopf $\mathcal{V}$-categories. (X nat. Frobenius)

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- natural isomorphisms $(M \circ N) \circ P \cong M \circ (N \circ P)$, $1_{\mathfrak{s}(M)} \circ M \cong M$, $M \circ 1_{\mathfrak{t}(M)} \cong M$ subject to axioms.

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0-cells, horizontal 1-cells and globular 2-morphisms make $\mathcal{H}(\mathbb{D})$.

Double category of enriched matrices

⋆ $\mathcal{V}$-**Mat** is the *horizontal bicategory* of a double category $\mathcal{V}$-$\mathbb{M}\mathbf{at}$.

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$$\begin{array}{ccc} X & \overset{S}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\;} & Y \\ f\downarrow & \Downarrow\alpha & \downarrow g \\ Z & \underset{T}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\;} & W \end{array} \quad \text{are } \alpha_{xy} \colon S_{x,y} \to T_{fx,gy}.$$

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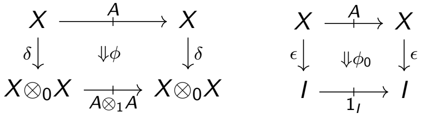

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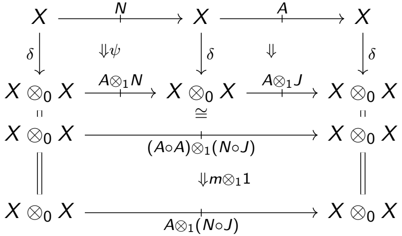

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⋆ Most interesting examples naturally arise in this framework

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$\mathbb{P}$**rof** (Pseudo)monoid $\mathcal{V}$ in **Cat** $\equiv$ monoidal category – (Hopf) bicomonad $\widehat{T} = \mathcal{V}(-, T-) \Leftrightarrow$ ordinary (Hopf) bicomonad T on (autonomous) monoidal $\mathcal{V}$

Examples in double context

$\mathcal{V}$-$\mathbb{M}$at Canonical structure $(X, \Delta, !)$ – bimonad on any X is a semi-Hopf $\mathcal{V}$-category, Hopf monad is a Hopf $\mathcal{V}$-category

$\mathbb{S}$pan Canonical structure $(X, \Delta, !)$ – bimonad on any X is a category, Hopf monad is a groupoid

$\mathbb{B}$Mod Monoid (A, μ, η) in $\mathbf{Alg}_R \equiv$ commutative algebra – bicomonad is A-bialgebroid, Hopf comonad is Hopf A-algebroid

$\mathbb{P}$rof (Pseudo)monoid $\mathcal{V}$ in $\mathbf{Cat} \equiv$ monoidal category – (Hopf) bicomonad $\widehat{T} = \mathcal{V}(-, T-) \Leftrightarrow$ ordinary (Hopf) bicomonad T on (autonomous) monoidal $\mathcal{V}$

Vertical (co)monoids in a double category determine new examples of Hopf bi(co)monads in its horizontal bicategory.

Thank you for your attention!