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Extremal and regular epimorphisms in the category $Equ\mathbb{E}$ of equivalence relations in a finitely complete category $\mathbb{E}$

Given a morphism of equivalence relations in $\mathbb{E}$:

$$\begin{tikzcd} R \arrow[rr, "\hat{f}"] \arrow[d, shift right=2, "d_0^R"'] \arrow[d, shift left=2, "d_1^R"] && S \arrow[d, shift right=2, "d_0^S"'] \arrow[d, shift left=2, "d_1^S"] \\ X \arrow[u] \arrow[rr, "f"'] && Y \arrow[u] \end{tikzcd}$$

it is clear that if the morphism $(f, \hat{f})$ is an extremal (resp. regular) epimorphim in the category $Equ\mathbb{E}$ of equivalence relations in $\mathbb{E}$, so is the morphism f in $\mathbb{E}$; but, in Set, there are extremal (resp. regular) epimorphims $(f, \hat{f})$ in Equ such that $\hat{f}$ is not. Actually, in any finitely complete category $\mathbb{E}$, given any morphism of split epimorphisms:

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, shift left, "f"] & X' \arrow[d, shift left, "f'"] \\ Y \arrow[u, shift left, "s"] \arrow[r, "y"'] & Y' \arrow[u, shift left, "s'"] \end{tikzcd}$$

the morphism $(x, R(x)) : R[f] \to R[f']$ between the induced kernel equivalence relations is extremal in $Equ\mathbb{E}$ as soon as so are the morphsims y and x in $\mathbb{E}$.
In this talk:

(1) we shall give a characterization of the extremal and regular epimorphisms in the category $Equ\mathbb{E}$ of equivalence relations in any finitely complete category $\mathbb{E}$,

(2) then we shall characterize the existence of the suprema of pairs of equivalence relations in term of the existence of some class Σ of extremal epimorphisms in $Equ\mathbb{E}$.

From that, it will appear that the context of the category $Equ\mathbb{E}$ of equivalence relations in $\mathbb{E}$ is a context which allows to exemplify the extreme variety of possible behaviours of extremal epimorphisms. Indeed:

(3) we shall characterize the congruence modular formula by the stability of the class Σ under pullback along a certain class of morphisms in $Equ\mathbb{E}$,

(4) then we shall characterize the situation where any extremal epimorphim in $Equ\mathbb{E}$ is stable under pullback, namely where the category $Equ\mathbb{E}$ is a regular category.

We shall exemplify the different situations by examples chosen among the varieties of Universal Algebra. Many aspects of the talk appeared in [1].

References:

[1] D. Bourn, Suprema of equivalence relations and non-regular Goursat categories, *Cahiers Top. Géom. Diff. Catég.* 59 (2018) 142–193.