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Towards a unification of Mal'tsev like categories

A Mal'tsev category can be defined in several equivalent ways. One possibility is to say that it is a category in which every relation is difunctional. A naturally Mal'tsev category can be defined as one in which every reflexive graph is the underlying graph of a unique groupoid structure. These two notions are well known and widely studied but there is still lacking a general result containing the most relevant features that are common to the two cases. Moreover, there is a third new case which shares some of the common properties and similarities with the other two. The notion introduced in [1], called weakly Mal'tsev category, is defined as a category with local products (pullbacks of split epimorphisms along split epimorphisms) in which every local product injection co-span is jointly epimorphic. Note the similarity with a well-known result by D. Bourn which characterizes Mal'tsev categories as those in which every local product injection cospan is jointly strongly epimorphic. In [2] it is proved that a category is weakly Mal'tsev if and only if every strong relation is difunctional (a strong relation is the same as a jointly strongly monomorphic span). Moreover, results in [3] show the similarity between the three notions of Mal'tsev like categories by dealing with appropriate classes of spans:
(a) $\mathcal{M}$ is the class of all spans — naturally Mal'tsev.
(b) $\mathcal{M}$ is the class of all relations — Mal'tsev categories.
(c) $\mathcal{M}$ is the class of all strong relations — weakly Mal'tsev categories.

The general result can be stated as follows. Let $\mathbb{C}$ be a category with pullbacks and equalizers and consider a class $\mathcal{M}$ of spans in $\mathbb{C}$ which contains all the identity spans and is stable under pullbacks. Denote by $F_i^{\mathcal{M}}$, with $i = 1, 2, 3, 4$, the restriction to the class $\mathcal{M}$ of the obvious forgetful functors, respectively, from pregroupoids to spans and from multiplicative graphs, internal categories, internal groupoids to reflexive graphs. The following conditions are equivalent:
(1) The functor $F_i^{\mathcal{M}}$ is an isomorphism, with $i = 1, 2, 3, 4$.
(2) The functor $F_i^{\mathcal{M}}$ has a section, with $i = 1, 2, 3, 4$.
(3) Every split square in $\mathbb{C}$ is $\mathcal{M}$-compatible.

The notion of split square compatible with a span is a new concept which will be introduced. It is, in some sense, a specialization to the notion of orthogonality between a span and a cospan.

References:

[1] N. Martins-Ferreira, *Weakly Mal'cev categories*, Theory Appl. Categ. **21** (6) (2008) 91–117.

[2] Z. Janelidze and N. Martins-Ferreira, *Weakly Mal'tsev categories and strong relations*, Theory Appl. Categ. **27** (5) (2012) 65–79.

[3] N. Martins-Ferreira and T. Van der Linden, *Categories vs. groupoids via generalised Mal'tsev properties*, Cah. Topol. Géom. Différ. Catég. **55** (2) (2014) 83–112.

*Joint work with Z. Janelidze and T. Van der Linden.