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A new characterisation of higher central extensions in semi-abelian categories

The concept of higher centrality is useful and unavoidable in the recent approach to homology and cohomology of non-abelian structures based on categorical Galois theory. In our work, higher central extensions are the covering morphisms with respect to certain Galois structures induced by a reflexion

$$\begin{tikzcd} \mathbb{X} \arrow[r, shift left, "\mathsf{Ab}"] \arrow[r, phantom, "\perp"] & \mathsf{Ab}(\mathbb{X}) \arrow[l, shift left, "\supset"] \end{tikzcd}$$

and can also be defined more generally, for any semi-abelian category $\mathbb{X}$ and any Birkhoff subcategory $\mathbb{B}$ of $\mathbb{X}$. The descriptions of higher central extensions in terms of algebraic conditions using "generalised commutators" is in general a non trivial problem. Through higher central extensions, in the presence of enough projectives, we know what is the complete description of the corresponding homology objects as higher Hopf formulae. For instance, in [4], the authors have proved that in any semi-abelian category with enough projectives which satisfy the *Smith is Huq* condition (SH), when considering the Birkhoff subcategory $\mathsf{Ab}(\mathbb{X})$ of all abelian objects in $\mathbb{X}$, via Janelidze's categorical Galois theory, the Hopf formulae take the following shape for any n-presentation F of the object Z

$$H_{n+1}(Z, \mathsf{Ab}(\mathbb{X})) \cong \frac{[F_n, F_n]_Q \wedge \bigwedge_{i \in n} K_i}{L_n[F]}$$

Here,
- the n-fold extension F may be viewed as an n-dimensional cube of arrows in $\mathbb{X}$;
- F_n is the initial object of F and the f_i are the initial arrows with kernel K_i;
- the bracket $[F_n, F_n]_Q$ is the Huq commutator of F_n with itself;
- the denominator object $L_n[F]$ is the smallest normal subobject of F_n, which, when divided out, makes F central. It is computed as the join of binary Huq or Higgins commutators

$$\bigvee_{I \subseteq n} [\bigwedge_{i \in I} K_i, \bigwedge_{i \in n \setminus I} K_i]_H$$

In general [1], since in any semi-abelian category, the object $L_n[F]$ is defined for any n-fold extension F and is such that F is central if and only if $L_n[F] = 0$, an n-fold extension F in any semi-abelian category with the *Smith is Huq* is central if and only if $\bigvee_{I \subseteq n} [\bigwedge_{i \in I} K_i, \bigwedge_{i \in n \setminus I} K_i]_H$ vanishes [4]. When we drop the *Smith is Huq* condition for semi-abelian categories, only double central extensions are characterised in terms of generalised commutators [2].

The aim of our work is to give a new characterisation of higher central extensions in semi-abelian categories [3, 1], one in terms of higher-order Higgins commutators [2], and

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which in contrast with [4] does not depend on the validity of the *Smith is Huq* condition. We study some properties of these higher-order Higgins commutators, and explore the connections with the categorical Galois theory of G. Janelidze. Given any n-fold extension F in any semi-abelian category, we study the object $L_n[F]$ and we prove that it take the following shape:

$$\bigvee_{I_0\cup\cdots\cup I_k=n, k\in\mathbb{N}^*}\big[\bigwedge_{i\in I_0} K_i,\dots,\bigwedge_{i\in I_k} K_i\big]_H$$

Here,
- $k\in\mathbb{N}^*$is an integer, I_l are subsets of n for all $0\leq l\leq k$ such that $I_0\cup\cdots\cup I_k=n$;
- the size of the higher-order Higgins commutator $[-,\cdots,-]_H$ stays bounded, and the join finite because the commutator in which an entry is repeated is smaller than the commutator with the repetition removed. By convention $\bigwedge_{\emptyset} K_i = F_n$.

As a consequence, in any semi-abelian category, an n-fold extension F is central if and only if the join of Higgins commutators

$$\bigvee_{I_0\cup\cdots\cup I_k=n, k\in\mathbb{N}^*}\big[\bigwedge_{i\in I_0} K_i,\dots,\bigwedge_{i\in I_k} K_i\big]_H$$

vanishes.

Hence, in the presence of enough projectives we obtain an explicit Hopf formula for the homology of Z with coefficients in the abelianisation functor.

$$H_{n+1}(Z,\mathsf{Ab}(\mathbb{X}))\cong\frac{[F_n,F_n]_Q\wedge\bigwedge_{i\in n}\ker(f_i)}{\bigvee_{I_0\cup\cdots\cup I_k=n, k\in\mathbb{N}^*}\big[\bigwedge_{i\in I_0} K_i,\cdots,\bigwedge_{i\in I_k} K_i\big]}$$

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