# Two factorizations of opfibrations between fibrations over a fixed base


Alan Cigoli

INdAM cofund Marie Curie fellow
Università degli Studi di Milano


(joint work with S. Mantovani and G. Metere)


CT 2018
Ponta Delgada, July 14, 2018

# Motivation

[Yoneda '60]

[Yoneda '60]

$\mathcal{A}$ abelian category.

# Motivation

[Yoneda '60]

$\mathcal{A}$ abelian category.

$$S \colon \mathbf{EXT}^n(\mathcal{A}) \to \mathcal{A} \times \mathcal{A}$$

$$(\ 0 \rightarrowtail B \to \cdots \to A \twoheadrightarrow 0\ ) \mapsto (A, B)$$

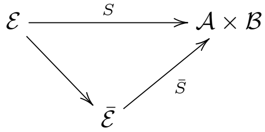
# Motivation

[Yoneda '60]

$\mathcal{A}$ abelian category.

$$S\colon \mathbf{EXT}^n(\mathcal{A}) \to \mathcal{A} \times \mathcal{A}$$
$$(\ 0 \twoheadrightarrow B \twoheadrightarrow \cdots \twoheadrightarrow A \twoheadrightarrow 0\ ) \mapsto (A, B)$$

Isolating the formal properties of this functor he was led to the notion of *regular span*, and he proved that such a functor admits a canonical factorization through a two-sided discrete fibration.



# Motivation

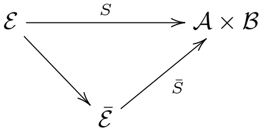
[Yoneda '60]

$\mathcal{A}$ abelian category.

$$S\colon \mathbf{EXT}^n(\mathcal{A}) \to \mathcal{A} \times \mathcal{A}$$

$$(\ 0 \twoheadrightarrow B \twoheadrightarrow \cdots \twoheadrightarrow A \twoheadrightarrow 0\ ) \mapsto (A, B)$$

Isolating the formal properties of this functor he was led to the notion of *regular span*, and he proved that such a functor admits a canonical factorization through a two-sided discrete fibration.




$\bar{\mathcal{E}}$ is obtained by taking connected components of each fiber of $S$.

Moving to a non-abelian context, symmetry is broken.

Moving to a non-abelian context, symmetry is broken.

Consider crossed extensions of groups.

$$P\colon \mathbf{XEXT}^n(\mathbf{Gp}) \to \mathbf{Mod}$$

$$(\ 0 \twoheadrightarrow B \twoheadrightarrow \cdots \twoheadrightarrow A \twoheadrightarrow 0\ ) \mapsto (A, B, \phi)$$

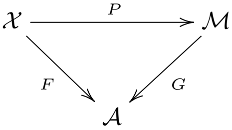
Moving to a non-abelian context, symmetry is broken.

Consider crossed extensions of groups.

$$P\colon \mathbf{XEXT}^n(\mathbf{Gp}) \to \mathbf{Mod}$$

$$(\ 0 \to B \to \cdots \to A \to 0\ ) \mapsto (A, B, \phi)$$

[C., Mantovani, Metere, Vitale '18]
In order to capture the properties of this functor, it is convenient to look at it as a fibered functor




whose restrictions $P_A\colon \mathcal{X}_A \to \mathcal{M}_A$ for each $A$ in $\mathcal{A}$ are opfibrations (*fiberwise opfibration*).

Moving to a non-abelian context, symmetry is broken.

Consider crossed extensions of groups.

$$P\colon \mathbf{XEXT}^n(\mathbf{Gp}) \to \mathbf{Mod}$$

$$(\ 0 \to B \to \cdots \to A \to 0\ ) \mapsto (A, B, \phi)$$

[C., Mantovani, Metere, Vitale '18]
In order to capture the properties of this functor, it is convenient to look at it as a fibered functor

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad P \quad} & \mathcal{M} \\ & {\scriptstyle F}\searrow \qquad \swarrow{\scriptstyle G} & \\ & \mathcal{A} & \end{array}$$

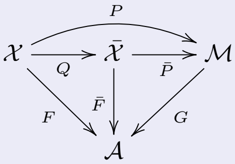
whose restrictions $P_A\colon \mathcal{X}_A \to \mathcal{M}_A$ for each $A$ in $\mathcal{A}$ are opfibrations (*fiberwise opfibration*).

In fact, any regular span is an instance of this:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad S \quad} & \mathcal{A} \times \mathcal{B} \\ & {\scriptstyle S_0}\searrow \qquad \swarrow{\scriptstyle Pr_0} & \\ & \mathcal{A} & \end{array}$$

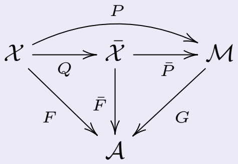
## Theorem

*Every fiberwise opfibration admits a universal factorization through a discrete opfibration* $\bar{P}$ *in* $\mathbf{Fib}(\mathcal{A})$.



## Theorem

*Every fiberwise opfibration admits a universal factorization through a discrete opfibration* $\bar{P}$ *in* $\mathbf{Fib}(\mathcal{A})$*.*




*This is nothing but the internal comprehensive factorization of* $P$ *in* $\mathbf{Fib}(\mathcal{A})$*.*

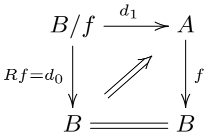
# Internal fibrations

Let $\mathcal{K}$ be a finitely complete 2-category.

# Internal fibrations

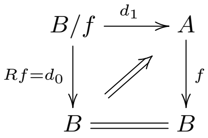
Let $\mathcal{K}$ be a finitely complete 2-category.

[Street '74]
One can define *internal fibrations* over an object $B$ in $\mathcal{K}$ as pseudo-algebras for the KZ-monad $R\colon \mathcal{K}/B \to \mathcal{K}/B$ defined by



Let $\mathcal{K}$ be a finitely complete 2-category.

[Street '74]
One can define *internal fibrations* over an object $B$ in $\mathcal{K}$ as pseudo-algebras for the KZ-monad $R\colon \mathcal{K}/B \to \mathcal{K}/B$ defined by




**Lemma**

*The identee of a morphism* $p\colon (A, f) \to (C, g)$ *in* $\mathbf{Fib}(B)$ *can be computed as in* $\mathcal{K}$*.*

Suppose that $\mathcal{K}$ has coinverters and coidentifiers and satisfies the hypothesis

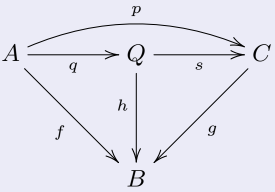
($\dagger$) For each $B$ in $\mathcal{K}$, the monad $R\colon \mathcal{K}/B \to \mathcal{K}/B$ preserves coinverters and coidentifiers.

Suppose that $\mathcal{K}$ has coinverters and coidentifiers and satisfies the hypothesis

(†) For each $B$ in $\mathcal{K}$, the monad $R\colon \mathcal{K}/B \to \mathcal{K}/B$ preserves coinverters and coidentifiers.

## Proposition

*Let $p\colon (A, f) \to (C, g)$ be a morphism in $\mathbf{Fib}(B)$ and $\omega$ its identee in $\mathcal{K}$. Then the coinverter (resp. coidentifier) $q$ of $\omega$ in $\mathcal{K}$ induces a factorization*




*of $p$ in $\mathbf{Fib}(B)$, where $q\colon (A, f) \to (Q, h)$ is the coinverter (resp. coidentifier) of $\omega$ in $\mathbf{Fib}(B)$.*

# Case study: **Cat**

# Case study: **Cat**

The property (†) follows from the fact that (op)fibrations are exponentiable.

## Case study: **Cat**

The property (†) follows from the fact that (op)fibrations are exponentiable. Moreover

### Proposition

*Each fibration $f\colon A \to B$ in* **Cat** *admits a factorization given by the coinverter of the identee of $f$ followed by a fibration in groupoids. This factorization coincides with the one given by (iterated coinverter, conservative functor).*

# Case study: **Cat**

The property (†) follows from the fact that (op)fibrations are exponentiable. Moreover

## Proposition

*Each fibration $f\colon A \to B$ in* **Cat** *admits a factorization given by the coinverter of the identee of $f$ followed by a fibration in groupoids. This factorization coincides with the one given by (iterated coinverter, conservative functor).*

This follows from two facts:

- an isofibration is conservative if and only if its identee is an iso (holds internally);
- split fibrations admit a reflection into split fibrations in groupoids.

## Case study: **Cat**

The property (†) follows from the fact that (op)fibrations are exponentiable. Moreover

### Proposition

*Each fibration $f\colon A \to B$ in* **Cat** *admits a factorization given by the coinverter of the identee of $f$ followed by a fibration in groupoids. This factorization coincides with the one given by (iterated coinverter, conservative functor).*

This follows from two facts:

- an isofibration is conservative if and only if its identee is an iso (holds internally);
- split fibrations admit a reflection into split fibrations in groupoids.

### Proposition

*Let $f\colon A \to B$ be a fibration in* **Cat**. *The comprehensive factorization of $f$ is given by the coidentifier of the identee of $f$ followed by the unique comparison functor to $f$, and this factorization coincides with the one given by (iterated coidentifier, discrete functor).*

## From **Cat** to **Fib**$(B)$

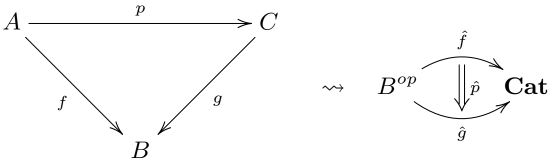
# From **Cat** to $\mathbf{Fib}(B)$

The two factorization systems above lift to $\mathbf{Fib}(B)$ for each $B$ in **Cat**.

# From **Cat** to **Fib**($B$)

The two factorization systems above lift to **Fib**($B$) for each $B$ in **Cat**.

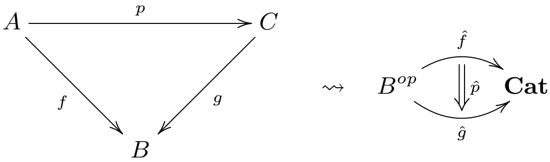
To see how this works, it is convenient to view fibrations over $B$ as pseudo-functors $[B^{op}, \mathbf{Cat}]$:



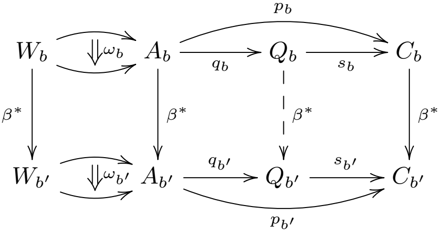
# From **Cat** to **Fib**$(B)$

The two factorization systems above lift to **Fib**$(B)$ for each $B$ in **Cat**.

To see how this works, it is convenient to view fibrations over $B$ as pseudo-functors $[B^{op}, \mathbf{Cat}]$:

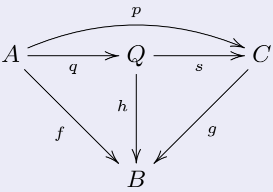



Then, for each $\beta \colon b' \to b$ in $B$



## Proposition

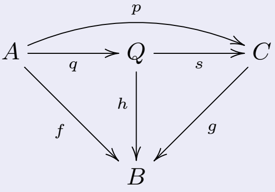
*Every fiberwise opfibration* $p\colon (A,f) \to (C,g)$ *in* $\mathbf{Fib}(B)$ *admits a comprehensive factorization*




*where* $q$ *is the coidentifier of the identee of* $p$ *and* $s$ *is a discrete opfibration in* $\mathbf{Fib}(B)$.

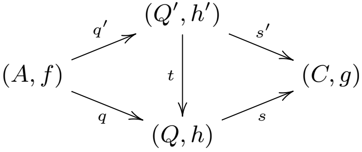
## Proposition

*Every fiberwise opfibration $p\colon (A, f) \to (C, g)$ in $\mathbf{Fib}(B)$ admits a comprehensive factorization*




*where $q$ is the coidentifier of the identee of $p$ and $s$ is a discrete opfibration in $\mathbf{Fib}(B)$.*

A similar result holds with $q'$ the coinverter of the identee of $p$ and $s'$ a fiberwise opfibration in groupoids. These two factorizations admit a unique comparison



# References


- A. S. Cigoli, S. Mantovani, G. Metere, and E. M. Vitale, Fibered aspects of Yoneda's regular spans, preprint arXiv: 1806.02376.
- R. Street and R. F. C. Walters, The comprehensive factorization of a functor, Bull. Amer. Math. Soc. 79 (5) (1973), 936–941.
- R. Street, Fibrations and Yoneda's lemma in a 2-category, in Category Seminar, Sydney 1972/73, Lecture Notes in Math. vol. 420, Springer-Verlag, 1974, 104–133.
- N. Yoneda, On Ext and exact sequences. J. Fac. Sci. Univ. Tokyo Sect. I 8 (1960) 507–576.