

Higher commutator conditions for extensions in Mal'tsev categories

Category Theory 2018 - Ponta Delgada

Arnaud Duvieusart

FNRS Research Fellow - Université catholique de Louvain

11th July 2018

(Joint work with Marino Gran)

Central extensions of groups

A group X is abelian iff

$$[X, X] = \{1\}.$$

A surjective epimorphism $f : X \rightarrow Y$ is a central extension iff

$$[\text{Ker}(f), X] = \{1\}.$$

Central extensions of groups

A group X is abelian iff

$$[X, X] = \{1\}.$$

A surjective epimorphism $f : X \rightarrow Y$ is a central extension iff

$$[\text{Ker}(f), X] = \{1\}.$$

Both of these concepts have been generalized to other situations:

- Categorical Galois Theory : General notion of central extension relatively to a reflective subcategory (Janelidze, Kelly, 1994[9])

Central extensions of groups

A group X is abelian iff

$$[X, X] = \{1\}.$$

A surjective epimorphism $f : X \rightarrow Y$ is a central extension iff

$$[\text{Ker}(f), X] = \{1\}.$$

Both of these concepts have been generalized to other situations:

- Categorical Galois Theory : General notion of central extension relatively to a reflective subcategory (Janelidze, Kelly, 1994[9])
- Commutators : Commutator of equivalence relations for Mal'tsev varieties (Smith, 1976) and exact Mal'tsev categories with coequalizers (Pedicchio, 1995 [11])

If \mathcal{C} is an exact Mal'tsev category with coequalizers, and $\mathbf{Ab}(\mathcal{C})$ is the subcategory of objects such that $[\nabla_X, \nabla_X] = \Delta_X$, then :

Proposition (Janelidze, Kelly, 2000[10]; Gran, 2004 [6])

An extension $f : X \rightarrow Y$ is central with respect to $\mathbf{Ab}(\mathcal{C})$ if and only if

$$[Eq[f], \nabla_X] = \Delta_X.$$

Proposition (Janelidze, 1991 [8]; Gran, Rossi, 2004[7]; Everaert, Van der Linden, 2010[5])

A double extension

$$\begin{array}{ccc}
 X & \xrightarrow{g} & Z \\
 f \downarrow & & \downarrow h \\
 Y & \xrightarrow{j} & W
 \end{array}$$

is central with respect to $\mathbf{Ab}(\mathcal{C})$ if and only if

$$[Eq[f] \wedge Eq[g], \nabla_X] = \Delta_X = [Eq[f], Eq[g]].$$

Reflexive graphs and internal groupoids

Let B be a fixed object of \mathcal{C} .

A reflexive graph $\mathbb{X} = (X, B, c, d, e)$ in \mathcal{C} is a diagram

$$X \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} B$$

such that $c \circ e = 1_B = d \circ e$.

Reflexive graphs and internal groupoids

Let B be a fixed object of \mathcal{C} .

A reflexive graph $\mathbb{X} = (X, B, c, d, e)$ in \mathcal{C} is a diagram

$$X \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} B$$

such that $c \circ e = 1_B = d \circ e$.

An internal groupoid is a reflexive graph endowed with a multiplication $\mu : X \times_B X \rightarrow X$ that is associative, unital, and has an inverse.

$$X \times_B X \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\mu} \\ \xrightarrow{\pi_2} \end{array} X \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} B$$

\uparrow
 σ

Reflexive graphs and internal groupoids

Let B be a fixed object of \mathcal{C} .

A reflexive graph $\mathbb{X} = (X, B, c, d, e)$ in \mathcal{C} is a diagram

$$X \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} B$$

such that $c \circ e = 1_B = d \circ e$.

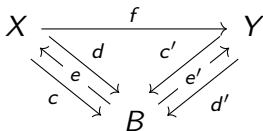
An internal groupoid is a reflexive graph endowed with a multiplication $\mu : X \times_B X \rightarrow X$ that is associative, unital, and has an inverse.

$$X \times_B X \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\mu} \\ \xrightarrow{\pi_2} \end{array} X \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} B$$

\uparrow
 σ

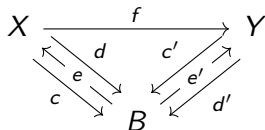
In a Mal'tsev category, a reflexive graph can have at most one structure of internal groupoid.

A morphism of reflexive graph is an arrow



making the triangles commute.

A morphism of reflexive graph is an arrow



making the triangles commute.

In a Mal'tsev category, $\mathbf{Grpd}(\mathcal{C})/B$ can be seen as a full subcategory of $\mathbf{RG}(\mathcal{C})/B$.

When \mathcal{C} is an exact Mal'tsev category with coequalizers, a reflexive graph (X, B, c, d, e) is an internal groupoid if and only if $[Eq[c], Eq[d]] = \Delta_X$.

When \mathcal{C} is an exact Mal'tsev category with coequalizers, a reflexive graph (X, B, c, d, e) is an internal groupoid if and only if $[Eq[c], Eq[d]] = \Delta_X$.
But can we also characterize the central extensions using a commutator condition?

Mal'tsev categories

A category is a Mal'tsev category if every reflexive relation is an equivalence relation.

Proposition (Carboni, Lambek, Pedicchio, 1991 [1])

If \mathcal{C} is a regular category, the following are equivalent:

- \mathcal{C} is Mal'tsev.
- $R \circ S = S \circ R$ for any $R, S \in \text{Eq}_X(\mathcal{C})$.

If \mathcal{C} is a variety, then this is equivalent to the existence of a ternary operation p satisfying $p(x, y, y) = x$ and $p(y, y, z) = z$.

Examples : **Grp** ($p(x, y, z) = xy^{-1}z$), **Rng**, **Lie**, any abelian category, **Grp(Top)**

Centralizing relations

\mathcal{C} regular Mal'tsev category, R, S equivalence relations on X .

Centralizing relations

\mathcal{C} regular Mal'tsev category, R, S equivalence relations on X .
A double equivalence relation is a diagram

$$\begin{array}{ccc}
 C & \begin{array}{c} \xrightarrow{q_1} \\ \xrightarrow{q_2} \end{array} & S \\
 \begin{array}{c} \downarrow p_1 \\ \downarrow p_2 \end{array} & & \begin{array}{c} \downarrow s_1 \\ \downarrow s_2 \end{array} \\
 R & \begin{array}{c} \xrightarrow{r_1} \\ \xrightarrow{r_2} \end{array} & X,
 \end{array}$$

where (C, p_1, p_2) is an equivalence relation on R , (C, q_1, q_2) is an equivalence relation on S , and $r_i p_j = s_j q_i$ for all $i, j \in \{1, 2\}$.

Centralizing relations

\mathcal{C} regular Mal'tsev category, R, S equivalence relations on X .
A double equivalence relation is a diagram

$$\begin{array}{ccc}
 C & \begin{array}{c} \xrightarrow{q_1} \\ \xrightarrow{q_2} \end{array} & S \\
 \begin{array}{c} \downarrow p_1 \\ \downarrow p_2 \end{array} & & \begin{array}{c} \downarrow s_1 \\ \downarrow s_2 \end{array} \\
 R & \begin{array}{c} \xrightarrow{r_1} \\ \xrightarrow{r_2} \end{array} & X,
 \end{array}$$

where (C, p_1, p_2) is an equivalence relation on R , (C, q_1, q_2) is an equivalence relation on S , and $r_i p_j = s_j q_i$ for all $i, j \in \{1, 2\}$.

A double equivalence relation is *centralizing* if any of these commutative squares are pullbacks.

Centrality and commutators

When \mathcal{C} is a regular Mal'tsev category, the following conditions are equivalent (Carboni-Pedicchio-Pirovano, [2]); we say that R, S centralize each other.

- R and S have a (unique) centralizing relation;
- there exists a (unique) *connector* $p : R \times_X S \rightarrow X$, satisfying

$$p(x, y, y) = x \text{ and } p(y, y, z) = z.$$

Centrality and commutators

When \mathcal{C} is a regular Mal'tsev category, the following conditions are equivalent (Carboni-Pedicchio-Pirovano, [2]); we say that R, S centralize each other.

- R and S have a (unique) centralizing relation;
- there exists a (unique) *connector* $p : R \times_X S \rightarrow X$, satisfying

$$p(x, y, y) = x \text{ and } p(y, y, z) = z.$$

When \mathcal{C} is exact and has coequalizers, one can define a commutator of equivalence relations (Pedicchio, 1995 [11]) such that R, S centralize each other if and only if $[R, S] = \Delta_X$.

In fact $[R, S]$ is the smallest equivalence relation whose coequalizer q is such that $q(R)$ and $q(S)$ centralize each other.

An internal reflexive graph $\mathbb{X} = (X, B, c, d, e)$ is an internal groupoid if and only if $[Eq[c], Eq[d]] = \Delta_X$.

An internal reflexive graph $\mathbb{X} = (X, B, c, d, e)$ is an internal groupoid if and only if $[Eq[c], Eq[d]] = \Delta_X$.

Proposition (Pedicchio, 1995 [11])

The category $\mathbf{Grpd}(\mathcal{C})/B$ is reflective in $\mathbf{RG}(\mathcal{C})/B$, with reflection given by

$$\begin{array}{ccc}
 X & \xrightarrow{\eta_{\mathbb{X}}} & \frac{X}{[Eq[c], Eq[d]]} \\
 \begin{array}{l} \swarrow d \\ \searrow e \\ \swarrow c \end{array} & & \begin{array}{l} \swarrow \bar{c} \\ \searrow \bar{e} \\ \swarrow \bar{d} \end{array} \\
 & & B
 \end{array}$$

It is also closed under subobjects and quotients, hence a *Birkhoff subcategory*.

The Galois structure

Let us denote \mathcal{E} the class of regular epimorphisms in $\mathbf{RG}(\mathcal{C})/B$. Then, since \mathcal{C} is regular:

- 1 any isomorphism is in \mathcal{E} ;
- 2 \mathcal{E} is pullback-stable;
- 3 \mathcal{E} is closed under composition.

Moreover, the reflector I preserves \mathcal{E} .

$\Gamma = (\mathbf{RG}(\mathcal{C})/B, \mathbf{Grpd}(\mathcal{C})/B, \mathcal{E}, I)$ is a Galois structure.

In this context we have another adjunction for every reflexive graph \mathbb{X} :

$$\mathbf{RG}(\mathcal{C})/B \downarrow_{\varepsilon} \mathbb{X} \begin{array}{c} \xrightarrow{I} \\ \perp \\ \xleftarrow{\eta_{\mathbb{X}}^*} \end{array} \mathbf{Grpd}(\mathcal{C})/B \downarrow_{\varepsilon} I(\mathbb{X}).$$

The Galois structure is admissible when $\eta_{\mathbb{X}}^*$ is fully faithful for all \mathbb{X} .

An extension $f : \mathbb{X} \rightarrow \mathbb{Y}$ is :

- Γ -trivial if it is in the image of $\eta_{\mathbb{Y}}^*$, or, equivalently, if

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\eta_{\mathbb{X}}} & I(\mathbb{X}) \\ f \downarrow & & \downarrow I(f) \\ \mathbb{Y} & \xrightarrow{\eta_{\mathbb{Y}}} & I(\mathbb{Y}) \end{array}$$

is a pullback.

- Γ -central if it is split by an extension, i.e. there is a pullback

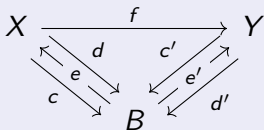
$$\begin{array}{ccc} \mathbb{Z} \times_{\mathbb{Y}} \mathbb{X} & \xrightarrow{g'} & \mathbb{X} \\ f' \downarrow & & \downarrow f \\ \mathbb{Z} & \xrightarrow{g} & \mathbb{Y} \end{array}$$

with f' trivial.

- Γ -normal if it splits itself, i.e. if the projections of its kernel pair are trivial.

Proposition (Everaert, Gran, 2006 [4])

Let \mathcal{V} be a Mal'tsev variety. An extension in $\mathbf{RG}(\mathcal{V})/B$

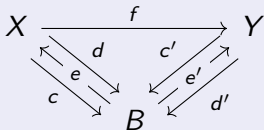


is central relatively to $\mathbf{Grpd}(\mathcal{V})/B$ if and only if

$$[Eq[f], Eq[c] \vee Eq[d]] = \Delta_X.$$

Theorem (Duvieusart, Gran, 2018 [3])

Let \mathcal{C} be an exact Mal'tsev category with coequalizers. An extension in $\mathbf{RG}(\mathcal{C})/B$



is central relatively to $\mathbf{Grpd}(\mathcal{C})/B$ if and only if

$$[Eq[f], Eq[c] \vee Eq[d]] = \Delta_X.$$

The class \mathcal{E} of regular epimorphism can also be seen as a full subcategory of $\mathbf{Arr}(\mathbf{RG}(\mathcal{C})/B)$. The class of central extensions then forms a reflective subcategory $\mathbf{CExt}(\mathbf{RG}(\mathcal{C})/B)$ of $\mathbf{Ext}(\mathbf{RG}(\mathcal{C})/B)$, with reflection given by

$$\begin{array}{ccc}
 X & \xrightarrow{q_{[Eq[c] \vee Eq[d], Eq[f]]}} & \frac{X}{[Eq[c] \vee Eq[d], Eq[f]]} \\
 & \searrow f & \swarrow \bar{f} \\
 & & Y.
 \end{array}$$

Definition

A *double extension* is a commutative square of regular epimorphisms

$$\begin{array}{ccc}
 \mathbb{X} & \xrightarrow{g} & \mathbb{Z} \\
 f \downarrow & & \downarrow h \\
 \mathbb{Y} & \xrightarrow{j} & \mathbb{W}
 \end{array}$$

such that the canonical arrow $\mathbb{X} \rightarrow \mathbb{Y} \times_{\mathbb{W}} \mathbb{Z}$ is a regular epimorphism in $\mathbf{RG}(\mathcal{C})/B$. Double extensions form a class \mathcal{E}_1 of arrows in the category $\mathbf{Ext}(\mathbf{RG}(\mathcal{C})/B)$.

Definition

A *double extension* is a commutative square of regular epimorphisms

$$\begin{array}{ccc}
 \mathbb{X} & \xrightarrow{g} & \mathbb{Z} \\
 f \downarrow & & \downarrow h \\
 \mathbb{Y} & \xrightarrow{j} & \mathbb{W}
 \end{array}$$

such that the canonical arrow $\mathbb{X} \rightarrow \mathbb{Y} \times_{\mathbb{W}} \mathbb{Z}$ is a regular epimorphism in $\mathbf{RG}(\mathcal{C})/B$. Double extensions form a class \mathcal{E}_1 of arrows in the category $\mathbf{Ext}(\mathbf{RG}(\mathcal{C})/B)$.

$\Gamma_1(\mathbf{Ext}(\mathbf{RG}(\mathcal{C})/B), \mathbf{CExt}(\mathbf{RG}(\mathcal{C})/B), l_1, \mathcal{E}_1)$ is again an admissible Galois structure. A double central extension is a double extension that is Γ_1 -central.

Theorem (Duvieusart, Gran, 2018 [3])

A double extension

$$\begin{array}{ccc}
 \mathbb{X} & \xrightarrow{g} & \mathbb{Z} \\
 f \downarrow & & \downarrow h \\
 \mathbb{Y} & \xrightarrow{j} & \mathbb{W}
 \end{array}$$

is central in $\mathbf{RG}(\mathcal{C})/B$ if and only if

$$[Eq[f], Eq[g]] = \Delta_X = [Eq[f] \wedge Eq[g], Eq[c] \vee Eq[d]].$$

Compact Hausdorff groups

$\mathcal{C} = \mathbf{Comp}(\mathbf{Grp})$ is semi-abelian, and satisfies the "Smith is Huq" condition.

Compact Hausdorff groups

$\mathcal{C} = \mathbf{Comp}(\mathbf{Grp})$ is semi-abelian, and satisfies the "Smith is Huq" condition.

Hence an extension

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \swarrow d & & \searrow c' \\
 & & B \\
 \swarrow e & & \searrow e' \\
 & & B \\
 \swarrow c & & \searrow d'
 \end{array}$$

in $\mathbf{RG}(\mathbf{Comp}(\mathbf{Grp}))/B$ is central if and only

$$\overline{[\text{Ker}[f], \text{Ker}[d] \cdot \text{Ker}[c]]} = \{1\}.$$

Similarly, a double extension

$$\begin{array}{ccc}
 X & \xrightarrow{g} & Z \\
 f \downarrow & & \downarrow h \\
 Y & \xrightarrow{j} & W
 \end{array}$$

is central if and only if

$$\overline{[Ker(f) \wedge Ker(g), Ker(c) \cdot Ker(d)]} = \{1\} = \overline{[Ker(f), Ker(g)]}.$$

Precrossed Lie Algebras

We have an equivalence of categories

$$\begin{array}{ccc}
 \mathbf{Grpd}(\mathbf{Lie})/B & \overset{\perp}{\rightleftarrows} & \mathbf{RG}(\mathbf{Lie})/B \\
 \updownarrow \cong & & \updownarrow \cong \\
 \mathbf{XMod}(\mathbf{Lie})/B & \overset{\perp}{\rightleftarrows} & \mathbf{PXMod}(\mathbf{Lie})/B
 \end{array}$$

Precrossed Lie Algebras

We have an equivalence of categories

$$\begin{array}{ccc}
 \mathbf{Grpd}(\mathbf{Lie})/B & \begin{array}{c} \xleftarrow{\perp} \\ \xrightarrow{\perp} \end{array} & \mathbf{RG}(\mathbf{Lie})/B \\
 \updownarrow \cong & & \updownarrow \cong \\
 \mathbf{XMod}(\mathbf{Lie})/B & \begin{array}{c} \xleftarrow{\perp} \\ \xrightarrow{\perp} \end{array} & \mathbf{PXMod}(\mathbf{Lie})/B
 \end{array}$$

This allows us to characterise central extensions in $\mathbf{PXMod}(\mathbf{Lie})/B$ with respect to $\mathbf{XMod}(\mathbf{Lie})/B$.

$$\begin{array}{ccc}
 L & \xrightarrow{f} & L' \\
 & \searrow \partial & \swarrow \partial' \\
 & & B
 \end{array}$$

We denote $\langle \text{Ker}(f), L \rangle$ the ideal of L generated by terms of the form

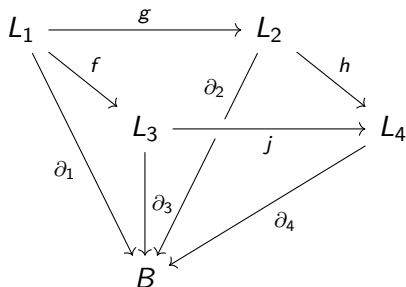
$$[k, l] \text{ or } \partial^{(l)}k, \quad k \in \text{Ker}(f), l \in L,$$

which we call the *Peiffer commutator*.

Then an extension is central if and only if

$$\langle \text{Ker}(f), L \rangle = 0.$$

Similarly, a double extension



is a double central extension if and only if

$$\langle \text{Ker}(f) \wedge \text{Ker}(g), X \rangle = 0 = [\text{Ker}(f), \text{Ker}(g)].$$

 A. Carboni, J. Lambek, and M. C. Pedicchio.

Diagram chasing in Mal'tsev categories.

J. Pure Appl. Algebra, 69(3):271–284, 1991.

 A. Carboni, M. C. Pedicchio, and N. Pirovano.

Internal graphs and internal groupoids in Mal'tsev categories.

In *Category theory 1991 (Montreal, PQ, 1991)*, CMS Conference Proceedings. AMS, Providence, RI, 1992.

 A. Duvieusart and M. Gran.

Higher commutator conditions for extensions in Mal'tsev categories.

To appear in *Journal of Algebra*.

 T. Everaert and M. Gran.

Precrossed modules and Galois theory.

J. Algebra, 297(1):292 – 309, 2006.



T. Everaert and T. Van der Linden.

A note on double central extensions in exact Mal'tsev categories.
Cah. Topol. Géom. Différ. Catég., 51(2):143–153, 2010.



M. Gran.

Applications of categorical Galois theory in universal algebra.
In *Galois theory, Hopf algebras, and semiabelian categories*, Fields
Inst. Commun. AMS, Providence, RI, 2004.



M. Gran and V. Rossi.

Galois theory and double central extensions.
Homology Homotopy Appl., 6(1):283–298, 2004.



G. Janelidze.

What is a double central extension?
Cahiers Topologie Géom. Différentielle Catég., 32(3):191–201, 1991.



G. Janelidze and G. M. Kelly.

Galois theory and a general notion of central extension.

J. Pure Appl. Algebra, 97(2):135 – 161, 1994.



G. Janelidze and G. M. Kelly.

Central extensions in Mal'tsev varieties.

Theory Appl. Categ., 7:No. 10, 219–226, 2000.



M. C. Pedicchio.

A categorical approach to commutator theory.

J. Algebra, 177(3):647 – 657, 1995.