Higher commutator conditions for extensions in Mal'tsev categories

Category Theory 2018 - Ponta Delgada

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(Joint work with Marino Gran)

Central extensions of groups

A group X is abelian iff

$$[X, X] = \{1\}.$$

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- Categorical Galois Theory : General notion of central extension relatively to a reflective subcategory (Janelidze,Kelly, 1994[9])
- Commutators : Commutator of equivalence relations for Mal'tsev varieties (Smith, 1976) and exact Mal'tsev categories with coequalizers (Pedicchio, 1995 [11])

If $\mathcal{C}$ is an exact Mal'tsev category with coequalizers, and $\mathbf{Ab}(\mathcal{C})$ is the subcategory of objects such that $[\nabla_X, \nabla_X] = \Delta_X$, then :

Proposition (Janelidze, Kelly, 2000[10]; Gran, 2004 [6])

An extension $f : X \to Y$ is central with respect to $\mathbf{Ab}(\mathcal{C})$ if and only if

$$[Eq[f], \nabla_X] = \Delta_X.$$

Proposition (Janelidze,1991 [8]; Gran, Rossi, 2004[7]; Everaert, Van der Linden, 2010[5])

A double extension

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[j]{} & W \end{array}$$

is central with respect to $\mathbf{Ab}(\mathcal{C})$ *if only if*

$$[Eq[f] \wedge Eq[g], \nabla_X] = \Delta_X = [Eq[f], Eq[g]].$$

Reflexive graphs and internal groupoids

Let B be a fixed object of $\mathcal{C}$.
A reflexive graph $\mathbb{X} = (X, B, c, d, e)$ in $\mathcal{C}$ is a diagram

$$X \underset{c}{\overset{d}{\underset{\longleftarrow e \longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}}} B$$

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Reflexive graphs and internal groupoids

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such that $c \circ e = 1_B = d \circ e$.
An internal groupoid is a reflexive graph endowed with a multiplication $\mu : X \times_B X \to X$ that is associative, unital, and has an inverse.

$$\begin{tikzcd} X \times_B X \arrow[r, shift left=3, "\pi_1"] \arrow[r, "\mu" description] \arrow[r, shift right=3, "\pi_2"'] & X \arrow[loop below, "\sigma"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B \arrow[l, "e" description] \end{tikzcd}$$

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In a Mal'tsev category, a reflexive graph can have at most one structure of internal groupoid.

A morphism of reflexive graph is an arrow

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, shift left=2, "d"] \arrow[dr, shift right=2, "c"'] & & Y \arrow[dl, shift right=2, "c'"'] \arrow[dl, shift left=2, "d'"] \\
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making the triangles commute.

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making the triangles commute.
In a Mal'tsev category, $\mathbf{Grpd}(\mathcal{C})/B$ can be seen as a full subcategory of $\mathbf{RG}(\mathcal{C})/B$.

When $\mathcal{C}$ is an exact Mal'tsev category with coequalizers, a reflexive graph (X, B, c, d, e) is an internal groupoid if and only if $[Eq[c], Eq[d]] = \Delta_X$.

When $\mathcal{C}$ is an exact Mal'tsev category with coequalizers, a reflexive graph (X, B, c, d, e) is an internal groupoid if and only if $[Eq[c], Eq[d]] = \Delta_X$. But can we also characterize the central extensions using a commutator condition?

Mal'tsev categories

A category is a Mal'tsev category if every reflexive relation is an equivalence relation.

Proposition (Carboni, Lambek, Pedicchio, 1991 [1])

If $\mathcal{C}$ is a regular category, the following are equivalent:

- *$\mathcal{C}$ is Mal'tsev.*
- *$R \circ S = S \circ R$ for any $R, S \in Eq_X(\mathcal{C})$.*

If $\mathcal{C}$ is a variety, then this is equivalent to the existence of a ternary operation p satisfying $p(x, y, y) = x$ and $p(y, y, z) = z$.

Examples : **Grp** ($p(x, y, z) = xy^{-1}z$), **Rng**, **Lie**, any abelian category, **Grp**(**Top**)

Centralizing relations

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A double equivalence relation is a diagram

$$\begin{tikzcd} C \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X, \end{tikzcd}$$

where (C, p_1, p_2) is an equivalence relation on R, (C, q_1, q_2) is an equivalence relation on S, and $r_i p_j = s_j q_i$ for all $i, j \in \{1, 2\}$.

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where (C, p_1, p_2) is an equivalence relation on R, (C, q_1, q_2) is an equivalence relation on S, and $r_i p_j = s_j q_i$ for all $i, j \in \{1, 2\}$.
A double equivalence relation is *centralizing* if any of these commutative squares are pullbacks.

Centrality and commutators

When $\mathcal{C}$ is a regular Mal'tsev category, the following conditions are equivalent (Carboni-Pedicchio-Pirovano, [2]); we say that R, S centralize each other.

- R and S have a (unique) centralizing relation;
- there exists a (unique) *connector* $p : R \times_X S \to X$, satisfying

$$p(x, y, y) = x \text{ and } p(y, y, z) = z.$$

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When $\mathcal{C}$ is exact and has coequalizers, one can define a commutator of equivalence relations (Pedicchio, 1995 [11]) such that R, S centralize each other if and only if $[R, S] = \Delta_X$.
In fact $[R, S]$ is the smallest equivalence relation whose coequalizer q is such that $q(R)$ and $q(S)$ centralize each other.

An internal reflexive graph $\mathbb{X} = (X, B, c, d, e)$ is an internal groupoid if and only if $[Eq[c], Eq[d]] = \Delta_X$.

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Proposition (Pedicchio, 1995 [11])

The category $\mathbf{Grpd}(\mathcal{C})/B$ *is reflective in* $\mathbf{RG}(\mathcal{C})/B$*, with reflection given by*

$$
\begin{tikzcd}
X \arrow[rr, "\eta_{\mathbb{X}}"] \arrow[dr, shift right=3, "c"'] \arrow[dr, shift left=1, "d"] & & \frac{X}{[Eq[c],Eq[d]]} \arrow[dl, shift right=1, "\overline{c}"'] \arrow[dl, shift left=3, "\overline{d}"] \\
& B \arrow[ul, "e"] \arrow[ur, "\overline{e}"] &
\end{tikzcd}
$$

It is also closed under subobjects and quotients, hence a *Birkhoff subcategory*.

The Galois structure

Let us denote $\mathcal{E}$ the class of regular epimorphisms in $\mathbf{RG}(\mathcal{C})/B$. Then, since $\mathcal{C}$ is regular:

1. any isomorphism is in $\mathcal{E}$;
2. $\mathcal{E}$ is pullback-stable;
3. $\mathcal{E}$ is closed under composition.

Moreover, the reflector I preserves $\mathcal{E}$.
$\Gamma = (\mathbf{RG}(\mathcal{C})/B, \mathbf{Grpd}(\mathcal{C})/B, \mathcal{E}, I)$ is a Galois structure.

In this context we have another adjunction for every reflexive graph $\mathbb{X}$:

$$\begin{tikzcd} \mathbf{RG}(\mathcal{C})/B \downarrow_{\mathcal{E}} \mathbb{X} \arrow[rr, bend left, "I"] \arrow[rr, phantom, "\perp"] & & \mathbf{Grpd}(\mathcal{C})/B \downarrow_{\mathcal{E}} I(\mathbb{X}). \arrow[ll, bend left, "\eta^*_X"] \end{tikzcd}$$

The Galois structure is admissible when $\eta^*_{\mathbb{X}}$ is fully faithful for all $\mathbb{X}$.

An extension $f : \mathbb{X} \to \mathbb{Y}$ is :

- Γ-*trivial* if it is in the image of $\eta^*_{\mathbb{Y}}$, or, equivalently, if

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\eta_{\mathbb{X}}"] \arrow[d, "f"'] & I(\mathbb{X}) \arrow[d, "I(f)"] \\ \mathbb{Y} \arrow[r, "\eta_{\mathbb{Y}}"'] & I(\mathbb{Y}) \end{tikzcd}$$

  is a pullback.
- Γ-*central* if it is split by an extension, i.e. there is a pullback

$$\begin{tikzcd} \mathbb{Z} \times_{\mathbb{Y}} \mathbb{X} \arrow[r, "g'"] \arrow[d, "f'"'] & \mathbb{X} \arrow[d, "f"] \\ \mathbb{Z} \arrow[r, "g"'] & \mathbb{Y} \end{tikzcd}$$

  with f' trivial.
- Γ-*normal* if it splits itself, i.e. if the projections of its kernel pair are trivial.

Proposition (Everaert, Gran, 2006 [4])

Let $\mathcal{V}$ be a Mal'tsev variety. An extension in $\mathbf{RG}(\mathcal{V})/B$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, shift left=2, "d"] \arrow[dr, shift right=2, "c"'] & & Y \arrow[dl, shift right=2, "c'"'] \arrow[dl, shift left=2, "d'"] \\
& B \arrow[ul, "e"] \arrow[ur, "e'"] &
\end{tikzcd}$$

is central relatively to $\mathbf{Grpd}(\mathcal{V})/B$ *if and only if*

$$[Eq[f], Eq[c] \vee Eq[d]] = \Delta_X.$$

Theorem (Duvieusart, Gran, 2018 [3])

Let $\mathcal{C}$ be an exact Mal'tsev category with coequalizers. An extension in $\mathbf{RG}(\mathcal{C})/B$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, shift left=2, "d"] \arrow[dr, shift right=2, "c"'] & & Y \arrow[dl, shift right=2, "c'"'] \arrow[dl, shift left=2, "d'"] \\
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\end{tikzcd}$$

is central relatively to $\mathbf{Grpd}(\mathcal{C})/B$ *if and only if*

$$[Eq[f], Eq[c] \vee Eq[d]] = \Delta_X.$$

The class $\mathcal{E}$ of regular epimorphism can also be seen as a full subcategory of $\mathbf{Arr}(\mathbf{RG}(\mathcal{C})/B)$. The class of central extensions then forms a reflective subcategory $\mathbf{CExt}(\mathbf{RG}(\mathcal{C})/B)$ of $\mathbf{Ext}(\mathbf{RG}(\mathcal{C})/B)$, with reflection given by

$$\begin{tikzcd} X \arrow[rr, "q_{[Eq[c]\vee Eq[d],Eq[f]]}"] \arrow[dr, "f"'] & & \frac{X}{[Eq[c]\vee Eq[d],Eq[f]]} \arrow[dl, "\overline{f}"] \\ & Y. & \end{tikzcd}$$

Definition

A *double extension* is a commutative square of regular epimorphisms

$$\begin{tikzcd} \mathbb{X} \arrow[r, "g"] \arrow[d, "f"'] & \mathbb{Z} \arrow[d, "h"] \\ \mathbb{Y} \arrow[r, "j"'] & \mathbb{W} \end{tikzcd}$$

such that the canonical arrow $\mathbb{X} \to \mathbb{Y} \times_{\mathbb{W}} \mathbb{Z}$ is a regular epimorphism in $\mathbf{RG}(\mathcal{C})/B$. Double extensions form a class $\mathcal{E}_1$ of arrows in the category $\mathbf{Ext}(\mathbf{RG}(\mathcal{C})/B)$.

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such that the canonical arrow $\mathbb{X} \to \mathbb{Y} \times_{\mathbb{W}} \mathbb{Z}$ is a regular epimorphism in $\mathbf{RG}(\mathcal{C})/B$. Double extensions form a class $\mathcal{E}_1$ of arrows in the category $\mathbf{Ext}(\mathbf{RG}(\mathcal{C})/B)$.

$\Gamma_1(\mathbf{Ext}(\mathbf{RG}(\mathcal{C})/B), \mathbf{CExt}(\mathbf{RG}(\mathcal{C})/B), I_1, \mathcal{E}^1)$ is again an admissible Galois structure. A double central extension is a double extension that is Γ_1-central.

Theorem (Duvieusart, Gran, 2018 [3])

A double extension

$$\begin{tikzcd} \mathbb{X} \arrow[r, "g"] \arrow[d, "f"'] & \mathbb{Z} \arrow[d, "h"] \\ \mathbb{Y} \arrow[r, "j"'] & \mathbb{W} \end{tikzcd}$$

is central in $\mathbf{RG}(\mathcal{C})/B$ *if and only if*

$$[Eq[f], Eq[g]] = \Delta_X = [Eq[f] \wedge Eq[g], Eq[c] \vee Eq[d]].$$

Compact Hausdorff groups

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Hence an extension

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, shift left=2, "d"] \arrow[dr, shift right=2, "c"'] & & Y \arrow[dl, shift right=2, "c'"'] \arrow[dl, shift left=2, "d'"] \\
& B \arrow[ul, "e"] \arrow[ur, "e'"'] &
\end{tikzcd}$$

in $\mathbf{RG}(\mathbf{Comp}(\mathbf{Grp}))/B$ is central if and only

$$\overline{[Ker[f], Ker[d] \cdot Ker[c]]} = \{1\}.$$

Similarly, a double extension

$$\begin{tikzcd} \mathbb{X} \arrow[r, "g"] \arrow[d, "f"'] & \mathbb{Z} \arrow[d, "h"] \\ \mathbb{Y} \arrow[r, "j"'] & \mathbb{W} \end{tikzcd}$$

is central if and only if

$$\overline{[Ker(f) \wedge Ker(g), Ker(c) \cdot Ker(d)]} = \{1\} = \overline{[Ker(f), Ker(g)]}.$$

Precrossed Lie Algebras

We have an equivalence of categories

$$\begin{tikzcd}
\mathbf{Grpd}(\mathbf{Lie})/B \arrow[r, hook] \arrow[r, phantom, "\perp"] \arrow[d, leftrightarrow, "\cong"] & \mathbf{RG}(\mathbf{Lie})/B \arrow[l, bend right] \arrow[d, leftrightarrow, "\cong"] \\
\mathbf{XMod}(\mathbf{Lie})/B \arrow[r, hook] \arrow[r, phantom, "\perp"] & \mathbf{PXMod}(\mathbf{Lie})/B \arrow[l, bend right]
\end{tikzcd}$$

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\mathbf{XMod}(\mathbf{Lie})/B \arrow[r, hook, "\perp"] & \mathbf{PXMod}(\mathbf{Lie})/B \arrow[l, bend right]
\end{tikzcd}$$

This allows us to characterise central extensions in $\mathbf{PXMod}(\mathbf{Lie})/B$ with respect to $\mathbf{XMod}(\mathbf{Lie})/B$.

$$\begin{tikzcd}
L \arrow[rr, "f"] \arrow[dr, "\partial"'] & & L' \arrow[dl, "\partial'"] \\
& B &
\end{tikzcd}$$

We denote $\langle Ker(f), L\rangle$ the ideal of L generated by terms of the form

$$[k, l] \text{ or } {}^{\partial(l)}k, \quad k \in Ker(f), l \in L,$$

which we call the *Peiffer commutator*.
Then an extension is central if and only if

$$\langle Ker(f), L\rangle = 0.$$

Similarly, a double extension

$$\begin{tikzcd}
L_1 \arrow[rr, "g"] \arrow[dr, "f"] \arrow[ddr, "\partial_1"'] & & L_2 \arrow[dr, "h"] \arrow[ddl, "\partial_2"'] & \\
& L_3 \arrow[rr, "j"'] \arrow[d, "\partial_3"] & & L_4 \arrow[dll, "\partial_4"] \\
& B & &
\end{tikzcd}$$

is a double central extension if and only if

$$\langle Ker(f) \wedge Ker(g), X \rangle = 0 = [Ker(f), Ker(g)].$$

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