

# Tensor topology

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**informatics**

## “Where things happen”

- ▶ Any monoidal category comes with built-in ‘space’
- ▶ Matches [examples](#)
- ▶ Universal notion of [support](#)
- ▶ [Completion](#) to actual space
- ▶ [Localisation](#) to subspaces
- ▶ [Embedding](#) separates out spatial dimension

See also

[Balmer, “Tensor triangular geometry”]

[Boyarchenko&Drinfeld, “Character sheaves of unipotent groups”]

## Idempotent subunits

Categorify central idempotents in ring:

$$\text{ISub}(\mathbf{C}) = \{ s: S \twoheadrightarrow I \mid S \otimes s: S \otimes S \rightarrow S \otimes I \text{ iso} \}$$

For most theory, split epic suffices  
For simplicity, let's take  $\mathbf{C}$  braided

## Example: order theory

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$$\begin{array}{ccc} \mathbf{Frame} & \begin{array}{c} \xrightarrow{\perp} \\ \xleftarrow{\text{ISub}} \end{array} & \mathbf{Quantale} \\ \{x \in Q \mid x^2 = x \leq 1\} & \xleftarrow{\quad} & Q \end{array}$$

## Example: logic

$$\begin{aligned} \text{ISub}(\text{Sh}(X)) &= \{S \vDash 1\} \\ &= \{S \subseteq X \mid S \text{ open}\} \in \mathbf{Frame} \end{aligned}$$

## Example: algebra

$$\text{ISub}(\mathbf{Mod}_R) = \{S \subseteq R \text{ ideal} \mid S = S^2 = \{x_1y_1 + \cdots + x_ny_n \mid x_i, y_i \in S\}\}$$

for nonunital bialgebra  $R$  in monoidal category



## Example: analysis

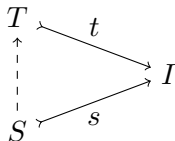
**Hilbert module** is  $C_0(X)$ -module with  $C_0(X)$ -valued inner product

$$C_0(X) = \{f: X \rightarrow \mathbb{C} \mid \forall \varepsilon > 0 \exists K \subseteq X: |f(X \setminus K)| < \varepsilon\}$$

$$\text{ISub}(\mathbf{Hilb}_{C_0(X)}) = \{S \subseteq X \text{ open}\}$$

## Semilattice

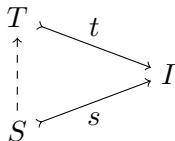
**Proposition:**  $\text{ISub}(\mathbf{C})$  is a semilattice,  $\wedge = \otimes$ ,  $1 = I$



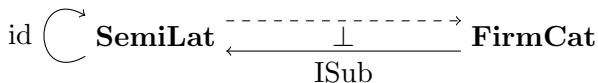
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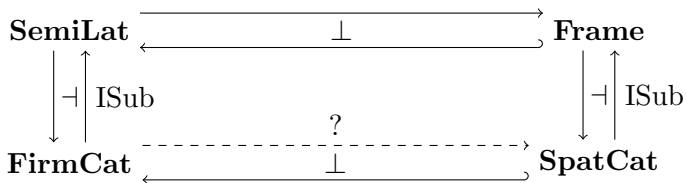


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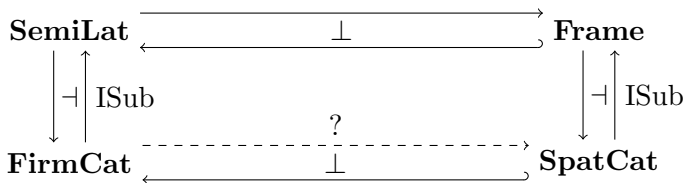
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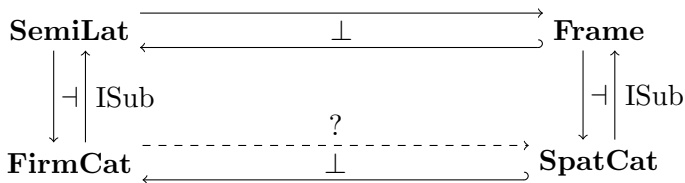
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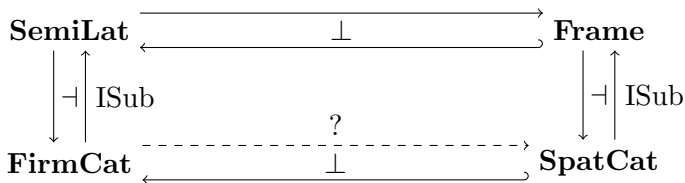
Idea:  $\widehat{\mathbf{C}} = [\mathbf{C}^{\text{op}}, \mathbf{Set}]$  is cocomplete

$$F \widehat{\otimes} G(A) = \int^{B,C} \mathbf{C}(A, B \otimes C) \times F(B) \times G(C)$$

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**Lemma:**  $\text{ISub}(\widehat{\mathbf{C}}, \widehat{\otimes})$  is frame, but  $\text{ISub}(\widehat{\mathbf{C}}) \neq \widehat{\text{ISub}(\mathbf{C})}$

# Support

Say  $s \in \text{ISub}(\mathbf{C})$  **supports**  $f: A \rightarrow B$  when

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \vdots & & \uparrow \simeq \\ B \otimes S & \xrightarrow{\quad B \otimes s \quad} & B \otimes I \end{array}$$



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 & \searrow F & \downarrow \widehat{F} \\
 & & Q \in \mathbf{Frame}
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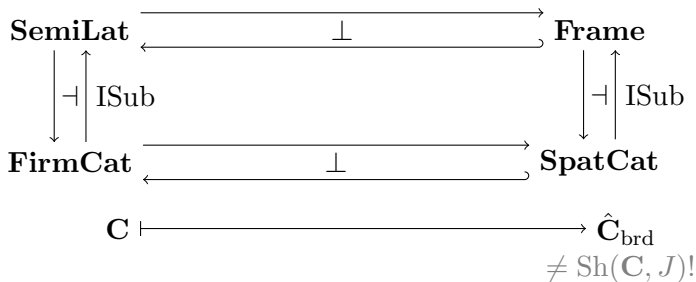
universal with  $F(f) = \bigvee \{F(s) \mid s \in \text{ISub}(\mathbf{C}) \text{ supports } f\}$

## Spatial completion

Call  $F: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$  **broad** when

$$F(A) \simeq \{(f, s): A \rightarrow B \mid s \in \text{supp}(f) \cap U\}$$

for some  $B \in \mathbf{C}$  and  $U \subseteq \text{ISub}(\mathbf{C})$ .



# Restriction

The full subcategory  $\mathbf{C}|_s$  of  $\mathbf{A}$  with  $A \otimes s$  invertible is:

- ▶ monoidal with tensor unit  $S$
- ▶ coreflective:  $\mathbf{C}|_s \begin{array}{c} \xrightarrow{\quad} \\ \dashv \perp \dashv \\ \xleftarrow{\quad} \end{array} \mathbf{C}$
- ▶ tensor ideal: if  $A \in \mathbf{C}$  and  $B \in \mathbf{C}|_s$ , then  $A \otimes B \in \mathbf{C}|_s$
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Examples:  $(\mathbf{Mod}_R)|_I = \mathbf{Mod}_I$ ,  $\text{Sh}(X)|_U = \text{Sh}(U)$

## Localisation

A **graded monad** is a monoidal functor  $\mathbf{E} \rightarrow [\mathbf{C}, \mathbf{C}]$

$$(\eta: A \rightarrow T(1), \mu: T(t) \circ T(s) \rightarrow T(s \otimes t))$$

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universal property of **localisation** for  $\Sigma_s = \{A \otimes s \mid A \in \mathbf{C}\}$

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**Lemma:**  $\Sigma = \{A \otimes s \mid A \in \mathbf{C}, s \in \text{ISub}(\mathbf{C})\}$  calculus of right fractions  
gives functor  $\mathbf{C} \rightarrow \text{Loc}(\mathbf{C}) = \mathbf{C}[\Sigma^{-1}]$  into **simple** category

## Support structure

Say  $\mathbf{C}$  is **slim** when any object is (domain of) idempotent subunit  
(Note:  $S$  determines  $s$ )

**Definition:** **support structure** is functor  $\zeta: \mathbf{C} \rightarrow \mathbf{C}$  with morphisms

- ▶  $\beta_A: \zeta(A) \rightarrow I$ ;
- ▶  $\gamma_A: A \rightarrow \zeta(A) \otimes A$ ;
- ▶  $\delta_A: \zeta(\zeta(A)) \rightarrow \zeta(A)$ ;

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**Proposition:**  $\delta_A$  is iso,  $\beta_A$  is idempotent,  $\zeta: \mathbf{C} \rightarrow \text{ISub}(\mathbf{C})$

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**Proposition:**  $\delta_A$  is iso,  $\beta_A$  is idempotent,  $\zeta: \mathbf{C} \rightarrow \text{ISub}(\mathbf{C})$

**Theorem:** Any supported monoidal category embeds into product of simple and slim one:  $\mathbf{C} \rightarrow \text{Loc}(\mathbf{C}) \times \text{ISub}(\mathbf{C})$

# Conclusion

- ▶ Any monoidal category comes with built-in ‘space’
- ▶ Matches *examples*
- ▶ Universal notion of *support*
- ▶ *Completion* to actual space
- ▶ *Localisation* to subspaces
- ▶ *Embedding* separates out spatial dimension

Further goals:

- ▶ Canonical status for support structure
- ▶ Dauns-Hofmann-like theorem
- ▶ Graphical calculus
- ▶ Applications: causality, concurrency

# Coherence

$$\begin{array}{ccc} \zeta^2 A & \xrightarrow{\delta} & \zeta A \\ \zeta\beta \downarrow & \searrow \beta & \downarrow \beta \\ \zeta I & \xrightarrow{\beta} & I \end{array}$$

$$\begin{array}{ccc} & \zeta A \otimes A & \\ \nearrow \gamma & & \searrow \beta \otimes A \\ A & \xrightarrow{\quad} & I \otimes A \end{array}$$

$$\begin{array}{ccc} & I & \\ \nearrow \beta & & \searrow \gamma \\ \zeta I & \xrightarrow{\quad} & \zeta I \otimes I \end{array}$$

$$\begin{array}{ccc} & I \otimes \zeta^2 A \otimes \zeta A & \\ \nearrow \beta \otimes \gamma & & \searrow I \otimes \delta \otimes \zeta A \\ \zeta A \otimes \zeta A & \xrightarrow{\quad} & I \otimes \zeta \otimes \zeta A \end{array}$$

$$\begin{array}{ccc} \zeta A & \xrightarrow{\gamma} & \zeta^2 A \otimes \zeta A \\ \beta \downarrow & & \downarrow \beta \otimes \beta \\ I & \xrightarrow{\quad} & I \otimes I \end{array}$$

## Complements

Subunit is **split** when  $\text{id} \circlearrowleft S \begin{array}{c} \xrightarrow{s} \\ \dashleftarrow{\quad} \end{array} I$   
 $\text{SISub}(\mathbf{C})$  is a sub-semilattice of  $\text{ISub}(\mathbf{C})$   
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If  $\mathbf{C}$  has zero object,  $\text{ISub}(\mathbf{C})$  has least element  $0$   
 $s, s^\perp$  are **complements** if  $s \wedge s^\perp = 0$  and  $s \vee s^\perp = 1$



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**Proposition:** when  $\mathbf{C}$  has finite biproducts,  
then  $s, s^\perp \in \text{SISub}(\mathbf{C})$  are complements  
if and only if they are biproduct injections

**Corollary:** if  $\oplus$  distributes over  $\otimes$ ,  
then SISub( $\mathbf{C}$ ) is a **Boolean** algebra  
(universal property?)