Monoidal Grothendieck Construction

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Outline

1. Fibrations and indexed categories

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2. Monoidal Grothendieck Construction
3. Global and fibrewise monoidal structures

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4. Examples

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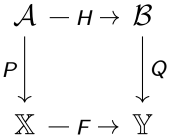

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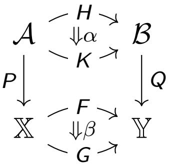

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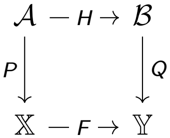

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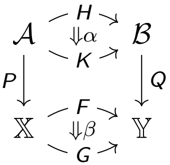

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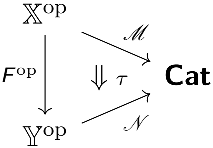

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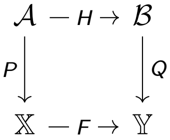

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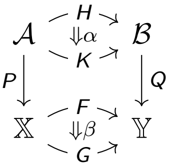

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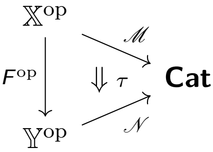

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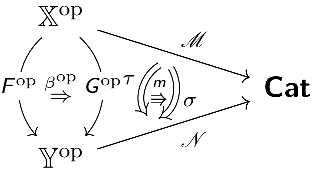

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The above lifts to a (cartesian) monoidal 2-equivalence $\mathbf{Fib} \simeq \mathbf{ICat}$.

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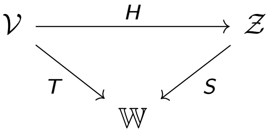

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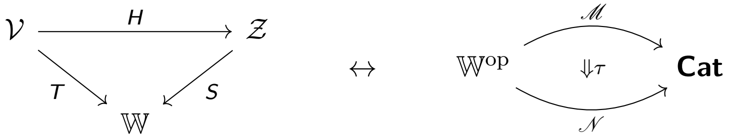

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If $\mathbb{X}$ is cartesian monoidal, all above structures are equivalent

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When $\mathbb{X}$ is cartesian, ‘monoidalness’ transfers from the target category to the structure of the functor and vice versa.

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▶Family fibration $\mathrm{Fam}(\mathcal{C})$ induced by the functor $[-,\mathcal{C}]\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$.
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⋆ Since **Set** is cartesian, these are both fibrewise monoidal as well.

Graphs and cospans

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The functor $F\colon \mathbf{Set} \to \mathbf{Cat}$ which maps any set X to $E \overset{s}{\underset{t}{\rightrightarrows}} X$, the category of all graphs with vertices X, induces opfibration $\mathbf{Grph} \to \mathbf{Set}$.

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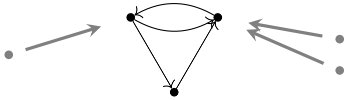

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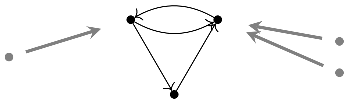

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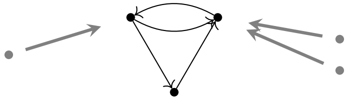

$\star$ Not only base $\mathbf{Set}$, but also total category is cocartesian (fibres too).

Network Models

[Baez, Foley, Moeller, Pollard]

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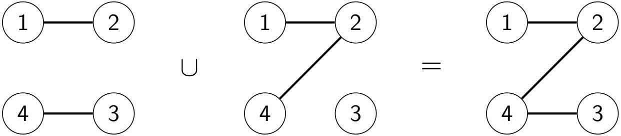

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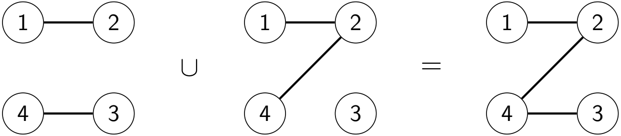

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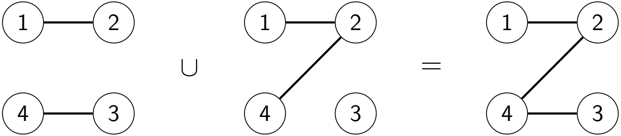

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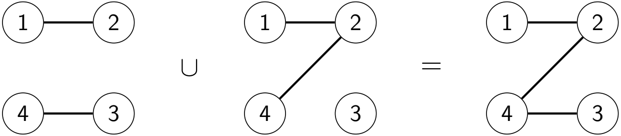

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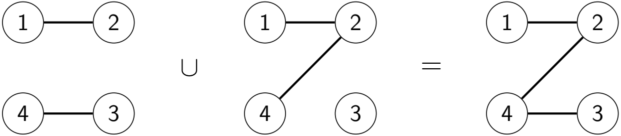

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⋆ Base $S(X)$ is not cocartesian; in many examples, it takes $+$ from **Set**.

Thank you for your attention!