Extremal and regular epimorphisms in the category $Equ\mathbb{E}$ of equivalence relations in a finitely complete category $\mathbb{E}$

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Outline

Internal equivalence relations

Extremal and regular epimorphims in $Equ\mathbb{E}$

Congruence modularity and distributivity

The regular context

Outline

Equivalence relations in $\mathbb{E}$ are represented with the simplicial notations; there is a left exact forgetful functor to the ground category:

$$\begin{tikzcd}
R \arrow[r, "\hat{f}"] \arrow[d, shift right=3, "d_0^R"'] \arrow[d, shift left=3, "d_1^R"] & S \arrow[d, shift right=3, "d_0^S"'] \arrow[d, shift left=3, "d_1^S"] & \\
X \arrow[u] \arrow[r, "f"'] & Y \arrow[u] & \mathit{Equ}\mathbb{E} \arrow[d, "(\ )_0"] \\
X \arrow[r, "f"'] & Y & \mathbb{E}
\end{tikzcd}$$

The functor $(\ )_0$ is a fibration whose fibers of are preorders

- what is equivalent to the fact that the functor $(\ )_0$ is faithful
- accordingly, given any diagram where R and S are equivalence relations:

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  there exist at most one map above f.
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- $\nabla(X)$ the undiscrete equiv. relation is the greatest element in the fibre above X
  while $\Delta(X)$ the discrete equiv. relation is the smallest one
- they give to the functor ( ) both a right and a left adjoint.
- A morphism of equivalence relation is called fibrant, when it is a discrete fibration:

$$\begin{array}{ccc} R & \xrightarrow{\hat{f}} & S \\ d_0^R \downarrow \uparrow \downarrow d_1^R & & d_0^S \downarrow \uparrow \downarrow d_1^S \\ X & \xrightarrow[f]{} & Y \end{array}$$

  i.e. when any of the squares in the diagram are pullbacks.

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One of the most important and classical result concerning equivalence relations is the following one:

Theorem

- *Given any split epimorphism* $(f, s) : X \rightleftarrows Y$*, the inverse image* $f^{-1} : Equ_Y\mathbb{E} \to Equ_X\mathbb{E}$ *induces a preorder bijection between:*
- *1) the equivalence relations on* Y
- *2) the equivalence relations on* X *containing* $R[f]$*.*
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Outline

We get the following characterizations in any finitely complete category $\mathbb{E}$:

Proposition

- *Given any extremal epimorphism* $(f, \hat{f}) : S \to T$ *in Equ*$\mathbb{E}$*, its underlying map* f *is an extremal epimorphism in* $\mathbb{E}$*.*
- *Suppose* f *is an extremal epimorphism in* $\mathbb{E}$*. The following conditions are equivalent in Equ*$\mathbb{E}$*:*
  1) the map $(f, \hat{f}) : S \to T$ *is extremal in Equ*$\mathbb{E}$
  2) the following diagram is a pushout:

$$\begin{tikzcd} \Delta_X \arrow[r, "\Delta_f"] \arrow[d, tail] & \Delta_Y \arrow[d, tail] \\ S \arrow[r, "(f,\hat{f})"'] & T \end{tikzcd}$$

  3) the map $(f, \hat{f}) : S \to T$ *is cocartesian with respect to the fibration* $(\ )_0$*.*
- *The extremal epimorphism* $(f, \hat{f}) : S \to T$ *is a regular epimorphism in Equ*$\mathbb{E}$ *if and only if its underlying map* $f : X \to Y$ *is a regular epimorphism in* $\mathbb{E}$*.*

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Here is the main observation of this talk: a characterization of the regular epimorphisms in $Equ\mathbb{E}$ above split epimorphisms:

- **Proposition**

  Given a split epimorphism $(g, t) : X \rightleftarrows Z$ in $\mathbb{E}$,
  a map $(g, \hat{g}) : R \to T$ above g is a regular epimorphism in $Equ\mathbb{E}$
  if and only if we have $g^{-1}(T) = R[g] \bigvee R$.

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$$\begin{tikzcd}
R \arrow[r, tail] \arrow[rr, bend left, "\hat{g}"] \arrow[dr, shift right=2, "d_0^R"'] \arrow[dr, shift left=2] & R[g] \bigvee R \arrow[r] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & T \arrow[l, dotted, shift left=2] \arrow[d, shift right=2, "d_0^T"'] \arrow[d, shift left=2, "d_1^T"] \\
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Proposition

Let (R, S) be any pair of equivalence relations on X in $\mathbb{E}$. TFAE:

- *1) the supremum $R \bigvee S$ does exist in Equ$\mathbb{E}$*
- *2) there is a cocartesian map (and hence a regular epimorphism) $(d_1^R, \bar{d}_1)$ in Equ$\mathbb{E}$ above the split epimorphism (d_1^R, s_0^R):*

$$\begin{tikzcd}
 & & S \arrow[ddl, shift right=2, "d_0^S"'] \arrow[ddl, shift left=2, "d_1^S"] \\
(d_0^R)^{-1}(S) \arrow[r, dotted, "\bar{d}_1"] \arrow[urr, bend left, "\check{d}_0^R"] \arrow[d, shift right=2, "\bar{\delta}_0^R"'] \arrow[d, shift left=2, "\bar{\delta}_1^R"] & W \arrow[d, shift right=2, "d_0^W"'] \arrow[d, shift left=2, "d_1^W"] & \\
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In this case we get: $W = R \bigvee S$, where W is the cocartesian image of $(d_0^R)^{-1}(S)$ along d_1^R.

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And the major result:

Theorem

The following conditions are equivalent:

- *1) Any pair (R, S) of equivalence relations has a supremum $R \bigvee S$*
- *2) Equ$\mathbb{E}$ has cocartesian maps with any domain above the split epimorphisms in $\mathbb{E}$.*
- *Under these assumptions, a morphism of equivalence relation is a regular epimorphism above the split epimorphism (f, s) in Equ$\mathbb{E}$:*

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if and only if we have $f^{-1}(V) = R[f] \bigvee S$.

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Basic situation where the fibration $(\ )_0$ is a cofibration as well:

Proposition

Suppose that a category $\mathbb{E}$ is such that the preorder determined by any fibre $Equ_Y\mathbb{E}$ has infima. Then the fibration $\mathcal{O}_{\mathbb{E}} : Equ\mathbb{E} \to \mathbb{E}$ is a cofibration as well.

- Accordingly it has regular epimorphisms with any domain above regular epimorphisms and a fortiori above split epimorphisms.; and consequently it has suprema of pairs of equiv. relations.

- **Proof.**
  Given any map $f : X \to Y$ and any equivalence relation R on X, set $f_!(R) = \bigwedge_{i \in I} T_i$ where

$$I = \{W;\ \text{equivalence relation on } Y / R \subset f^{-1}(W)\}$$

- Of course, it is the case for any variety of Universal Algebra.

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Congruence modularity:

$$(R \vee S) \wedge T) = R \vee (S \wedge T); \quad \text{provided that}: \ \mathrm{R} \subset \mathrm{T}$$

So, we get:

- **Proposition**

  Suppose $\mathbb{E}$ has suprema of pairs of equivalence relations. TFAE:

  1) $\mathbb{E}$ is congruence modular

  2) the cocartesian maps above split epimorphims are stable under pullbacks along maps in the fibers of $(\ )_0$.

  - Accordingly the categorical congruence modularity is a kind of part of the property of *Equ*$\mathbb{E}$ being regular; so that:
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  - Accordingly the categorical congruence modularity is a kind of part of the property of *Equ*$\mathbb{E}$ being regular; so that:
  - *Equ*$\mathbb{E}$ with suprema of pairs + *Equ*$\mathbb{E}$ regular category imply categorical congruence modularity.

Congruence modularity:

$$(R \vee S) \wedge T) = R \vee (S \wedge T); \text{ provided that : } \mathrm{R} \subset \mathrm{T}$$

So, we get:

- **Proposition**

  Suppose $\mathbb{E}$ has suprema of pairs of equivalence relations. TFAE:

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What is missing in order to make $Equ\mathbb{E}$ regular:

- **Pullback stability of cocartesian maps along cartesian maps in** $Equ\mathbb{E}$

Theorem

Suppose $\mathbb{E}$ has suprema of pairs of equivalence relations. TFAE:

1) the cocartesian maps above split epimorphism (f, s) are stable under pullbacks along cartesian maps in $Equ\mathbb{E}$

2) given any fibrant morphism $(g, \hat{g}) : R \to R'$ of equivalence relations with $g : X \to X'$ and any equivalence relation T on X', we get: $g^{-1}(R' \bigvee T) = R \bigvee g^{-1}(T)$.

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Outline

We shall suppose now that the ground category $\mathbb{E}$ is regular.

- It is well known that we can extend the result on inverse images from split epimorphisms to regular epimorphisms:

- **Theorem**
  When $\mathbb{E}$ is a regular category, given any regular epimorphism $f : X \twoheadrightarrow Y$, the inverse image $f^{-1} : Equ_Y\mathbb{E} \to Equ_X\mathbb{E}$ induces a preorder bijection between:

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From that, it is a very easy to extend the previous characterization to any regular epimorphisms in *Equ*$\mathbb{E}$:

Proposition

- *Given a regular category $\mathbb{E}$ and a regular epimorphism $g : X \twoheadrightarrow Z$, a map $(g, \hat{g}) : R \to T$ above g in Equ$\mathbb{E}$:*

$$\begin{tikzcd}
R \arrow[r, tail] \arrow[rr, bend left, "\hat{g}"] \arrow[dr, shift right=2, "d_0^R"'] \arrow[dr, shift left=2] & R[g]\bigvee R \arrow[r] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & T \arrow[d, shift right=2, "d_0^T"'] \arrow[d, shift left=2, "d_1^T"] \\
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So, we can also extend our theorem about the existence of suprema of pairs of equivalence relations in a very interesting formulation:

Proposition

- *Let $\mathbb{E}$ be a regular category. TFAE*
- *(i) Equ$\mathbb{E}$ has suprema of equivalence relations.*
- *(ii) Equ$\mathbb{E}$ has coequalizers of effective relations*
- *(iii) Equ$\mathbb{E}$ has cocartesian maps with any domain above any regular epimorphism in $\mathbb{E}$*
- *In this case a morphism $(f, \hat{f}) : S \to T$ is regular in Equ$\mathbb{E}$ if and only if f is regular in $\mathbb{E}$ and $f^{-1}(T) = R[f] \bigvee S$.*

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We are now in position to answer the question: when is $Equ\mathbb{E}$ a regular category?

Theorem

- *Given any category $\mathbb{E}$, the following conditions are equivalent:*
- *(i) the category $Equ\mathbb{E}$ is regular*
- *(ii) the category $\mathbb{E}$ is regular, cc-modular*
  and such that:
  (*) for any fibrant morphism $(g, \hat{g}) : R \to R'$
  and any equivalence relation S on the codomain Y of the map g
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From results from Janelidze, Marki Tholen, Ursini and Everaert, we know that any ideal determined variety $\mathbb{V}$ is such that *Equ*$\mathbb{V}$ is a regular category.

- By Raftery, we know that there are ideal dtermined variety which are not 3-permutable. For instance the variety of Lower BCK semi-lattices.
- Accordingly we know that there are varieties $\mathbb{V}$ such that:
  1) *Equ*$\mathbb{V}$ is a regular category
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