

Extremal and regular epimorphisms in the category $Equ\mathbb{E}$ of equivalence relations in a finitely complete category \mathbb{E}

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Outline

Internal equivalence relations

Extremal and regular epimorphisms in $Equ\mathbb{E}$

Congruence modularity and distributivity

The regular context

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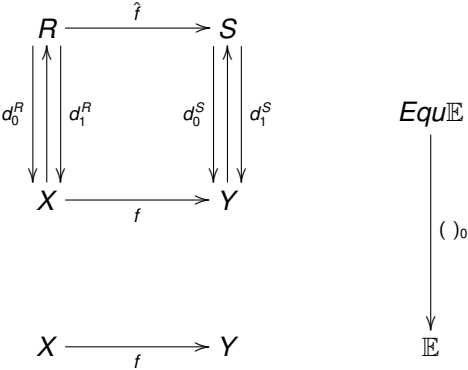
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Equivalence relations in \mathbb{E} are represented with the simplicial notations; there is a left exact forgetful functor to the ground category:



The functor $()_0$ is a fibration whose fibers of are **preorders**

- ▶ what is equivalent to the fact that the functor $()_0$ is **faithful**
- ▶ accordingly, given any diagram where R and S are equivalence relations:

$$\begin{array}{ccc}
 R & \xrightarrow{\hat{f}} & S \\
 d_0^R \downarrow \uparrow & & \downarrow \uparrow d_0^S \\
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there exist at most one map above f .
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there exist at most one map above f .

It is the case if and only if $R \subset f^{-1}(S)$.

- ▶ $\nabla(X)$ the undiscrete equiv. relation is the greatest element in the fibre above X
while $\Delta(X)$ the discrete equiv. relation is the smallest one
- ▶ they give to the functor $()$ both a right and a left adjoint.
- ▶ A morphism of equivalence relation is called **fibrant**, when it is a discrete fibration:

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 R & \xrightarrow{\hat{f}} & S \\
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One of the most important and classical result concerning equivalence relations is the following one:

Theorem

- ▶ Given any split epimorphism $(f, s) : X \rightrightarrows Y$, the inverse image $f^{-1} : Equ_Y \mathbb{E} \rightarrow Equ_X \mathbb{E}$ induces a preorder bijection between:
- ▶ 1) the equivalence relations on Y
- ▶ 2) the equivalence relations on X containing $R[f]$.
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We get the following characterizations in any finitely complete category \mathbb{E} :

Proposition

- ▶ Given any extremal epimorphism $(f, \hat{f}) : S \rightarrow T$ in $\text{Equ}\mathbb{E}$, its underlying map f is an extremal epimorphism in \mathbb{E} .
- ▶ Suppose f is an extremal epimorphism in \mathbb{E} . The following conditions are equivalent in $\text{Equ}\mathbb{E}$:
 - 1) the map $(f, \hat{f}) : S \rightarrow T$ is extremal in $\text{Equ}\mathbb{E}$
 - 2) the following diagram is a *pushout*:

$$\begin{array}{ccc} \Delta_X & \xrightarrow{\Delta_f} & \Delta_Y \\ \downarrow \gamma & & \downarrow \gamma \\ S & \xrightarrow{(f, \hat{f})} & T \end{array}$$

3) the map $(f, \hat{f}) : S \rightarrow T$ is cocartesian with respect to the fibration $(\)_0$.

- ▶ The extremal epimorphism $(f, \hat{f}) : S \rightarrow T$ is a regular epimorphism in $\text{Equ}\mathbb{E}$ if and only if its underlying map $f : X \rightarrow Y$ is a regular epimorphism in \mathbb{E} .

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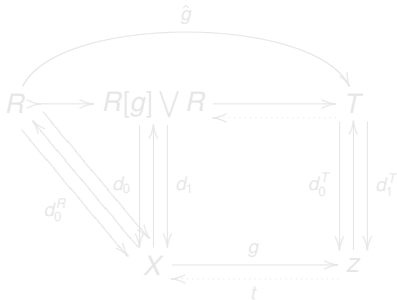
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Here is the **main observation** of this talk: a characterization of the regular epimorphisms in $Equ\mathbb{E}$ above split epimorphisms:

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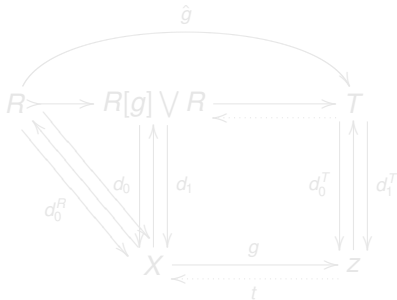
Given a split epimorphism $(g, t) : X \rightrightarrows Z$ in \mathbb{E} ,
 a map $(g, \hat{g}) : R \rightarrow T$ above g is **a regular epimorphism** in $Equ\mathbb{E}$
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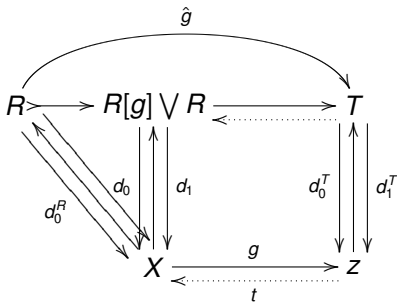
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Proposition

Let (R, S) be any pair of equivalence relations on X in \mathbb{E} . TFAE:

- ▶ 1) the supremum $R \vee S$ does exist in $\text{Equ}\mathbb{E}$
- ▶ 2) there is a cocartesian map (and hence a regular epimorphism) (d_1^R, \bar{d}_1) in $\text{Equ}\mathbb{E}$ above the split epimorphism (d_1^R, s_0^R) :

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 & & & & S \\
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 (d_0^R)^{-1}(S) & \cdots \rightarrow & W & & \\
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 R & \xrightarrow{d_1^R} & X & & \\
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In this case we get: $W = R \vee S$, where W is the cocartesian image of $(d_0^R)^{-1}(S)$ along d_1^R .

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And the major result:

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The following conditions are equivalent:

- ▶ 1) Any pair (R, S) of equivalence relations has a supremum $R \vee S$
- ▶ 2) $\text{Equ}\mathbb{E}$ has cocartesian maps with any domain above the split epimorphisms in \mathbb{E} .
- ▶ Under these assumptions, a morphism of equivalence relation is a regular epimorphism above the split epimorphism (f, s) in $\text{Equ}\mathbb{E}$:

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if and only if we have $f^{-1}(V) = R[f] \vee S$.

- ▶ We then set $V = f_!(S)$, for the *cocartesian image* of S along f .

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Basic situation where the fibration $()_0$ is a cofibration as well:

Proposition

Suppose that a category \mathbb{E} is such that the preorder determined by any fibre $Equ_Y \mathbb{E}$ has *infima*. Then the fibration $\mathcal{O}_{\mathbb{E}} : Equ \mathbb{E} \rightarrow \mathbb{E}$ is a *cofibration* as well.

- ▶ Accordingly it has regular epimorphisms with any domain above regular epimorphisms and a fortiori above split epimorphisms.; and consequently it has *suprema of pairs of equiv. relations*.
- ▶ **Proof.**

Given any map $f : X \rightarrow Y$ and any equivalence relation R on X , set $f_i(R) = \bigwedge_{i \in I} T_i$ where

$$I = \{W; \text{equivalence relation on } Y/R \subset f^{-1}(W)\}$$

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$$(R \vee S) \wedge T = R \vee (S \wedge T); \text{ provided that } : R \subset T$$

So, we get:

► Proposition

Suppose \mathbb{E} has suprema of pairs of equivalence relations. TFAE:

1) \mathbb{E} is congruence modular

2) the cocartesian maps above split epimorphisms are stable under pullbacks along maps in the fibers of $(\)_0$.

- Accordingly the **categorical congruence modularity** is a kind of part of the property of $Equ\mathbb{E}$ being regular; so that:
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What is missing in order to make $Equ\mathbb{E}$ regular:

- ▶ **Pullback stability of cocartesian maps along cartesian maps in $Equ\mathbb{E}$**

Theorem

Suppose \mathbb{E} has suprema of pairs of equivalence relations. TFAE:

1) the cocartesian maps above split epimorphism (f, s) are stable under pullbacks along cartesian maps in $Equ\mathbb{E}$

2) given any fibrant morphism $(g, \hat{g}) : R \rightarrow R'$ of equivalence relations with $g : X \rightarrow X'$ and any equivalence relation T on X' , we get: $g^{-1}(R' \vee T) = R \vee g^{-1}(T)$.

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$$\begin{array}{ccc}
 & g^{-1}(T) & \xrightarrow{\bar{g}} & T \\
 & \nearrow & & \nearrow \\
 R & \xrightarrow{\hat{g}} & R' & \\
 \downarrow d_0^R & & \downarrow d_0^{R'} & \\
 X & \xrightarrow{g} & X' & \\
 \uparrow d_1^R & & \uparrow d_1^{R'} & \\
 & \nwarrow & & \nwarrow
 \end{array}$$

The diagram illustrates a commutative square in the category $\mathbf{Equ}\mathbb{E}$. The top horizontal arrow is $\bar{g}: g^{-1}(T) \rightarrow T$. The bottom horizontal arrow is $g: X \rightarrow X'$. The left vertical arrow is $\hat{g}: R \rightarrow X$, and the right vertical arrow is $\hat{g}: R' \rightarrow X'$. The top-left corner is a square with arrows $d_0^R: R \rightarrow X$ (down) and $d_1^R: X \rightarrow R$ (up). The top-right corner is a square with arrows $d_0^{R'}: R' \rightarrow X'$ (down) and $d_1^{R'}: X' \rightarrow R'$ (up). The top-left and top-right corners are connected by diagonal arrows \nearrow and \nwarrow .

▶ (g, \hat{g}) fibrant in $\mathbf{Equ}\mathbb{E}$ implies $g^{-1}(R' \vee T) = R \vee g^{-1}(T)$.

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Outline

Internal equivalence relations

Extremal and regular epimorphisms in $Equ\mathbb{E}$

Congruence modularity and distributivity

The regular context

We shall suppose now that the ground category \mathbb{E} is regular.

- ▶ It is well known that we can extend the result on inverse images from split epimorphisms to regular epimorphisms:

▶ **Theorem**

When \mathbb{E} is a regular category, given any regular epimorphism $f : X \twoheadrightarrow Y$, the inverse image $f^{-1} : \text{Equ}_Y \mathbb{E} \rightarrow \text{Equ}_X \mathbb{E}$ induces a preorder bijection between:

- 1) the equivalence relations on Y*
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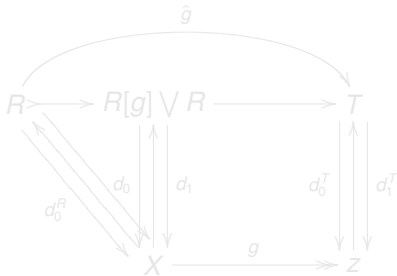
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From that, it is a very easy to extend the previous characterization to any regular epimorphisms in $Equ\mathbb{E}$:

Proposition

- ▶ Given a regular category \mathbb{E} and a regular epimorphism $g : X \rightarrow Z$, a map $(g, \hat{g}) : R \rightarrow T$ above g in $Equ\mathbb{E}$:

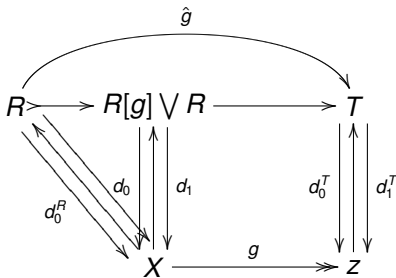


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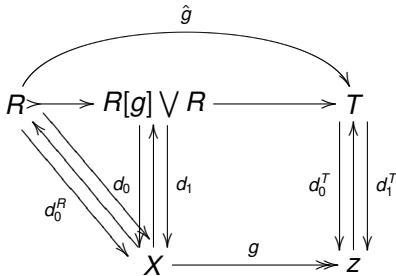


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So, we can also extend our theorem about the existence of suprema of pairs of equivalence relations in a very interesting formulation:

Proposition

- ▶ *Let \mathbb{E} be a regular category. TFAE*
- ▶ *(i) $\text{Equ}\mathbb{E}$ has suprema of equivalence relations.*
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- ▶ *(iii) $\text{Equ}\mathbb{E}$ has cocartesian maps with any domain above any regular epimorphism in \mathbb{E}*
- ▶ *In this case a morphism $(f, \hat{f}) : S \rightarrow T$ is regular in $\text{Equ}\mathbb{E}$ if and only if f is regular in \mathbb{E} and $f^{-1}(T) = R[f] \vee S$.*

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We are now in position to answer the question: **when is $\text{Equ}\mathbb{E}$ a regular category?**

Theorem

- ▶ Given any category \mathbb{E} , the following conditions are equivalent:
- ▶ (i) the category $\text{Equ}\mathbb{E}$ is regular
- ▶ (ii) the category \mathbb{E} is regular, cc-modular and such that:
(* for any fibrant morphism $(g, \hat{g}) : R \rightarrow R'$
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