Observations on the Shifting Lemma

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regular Mal'tsev categories

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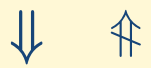

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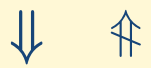

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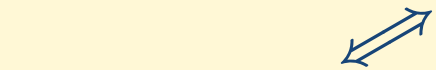

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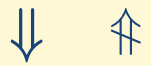

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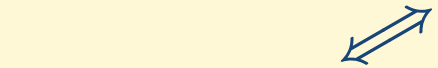

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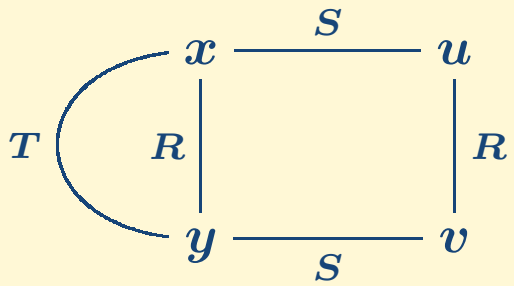

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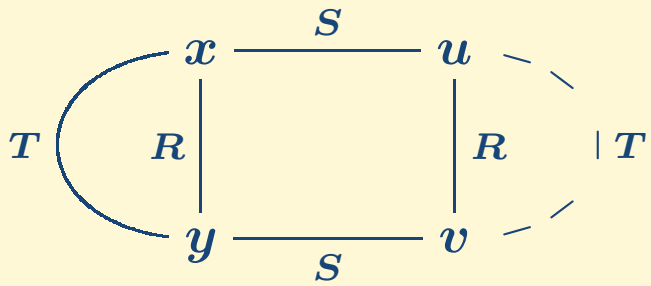

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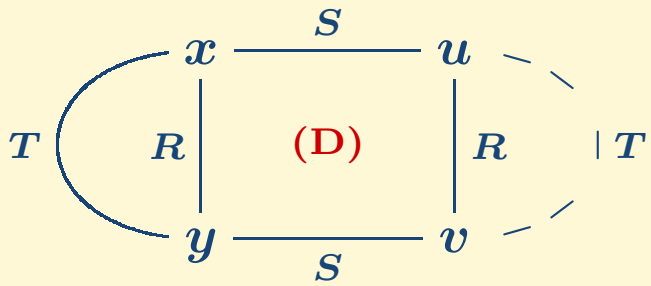

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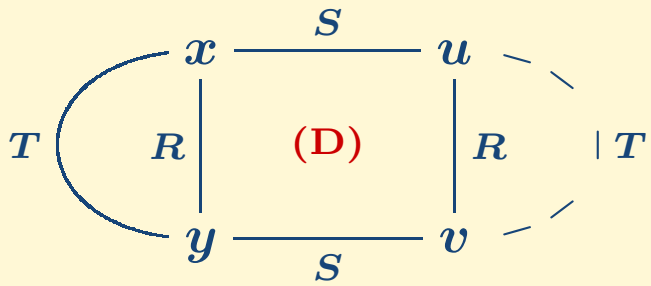

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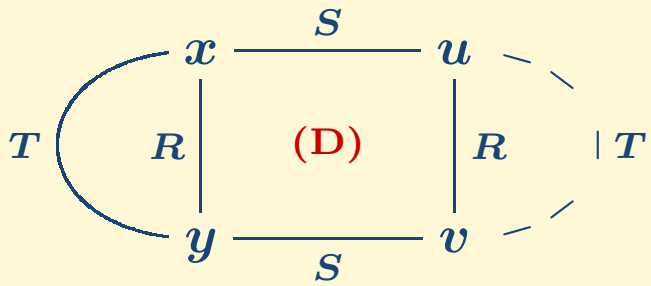

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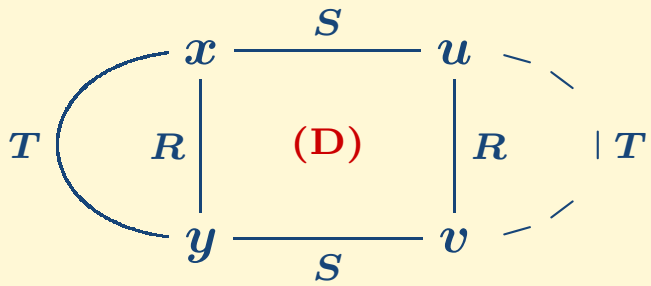

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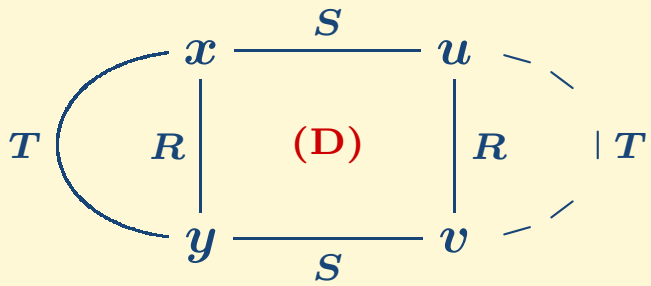

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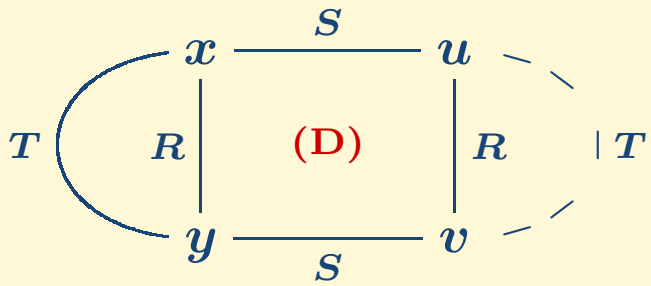

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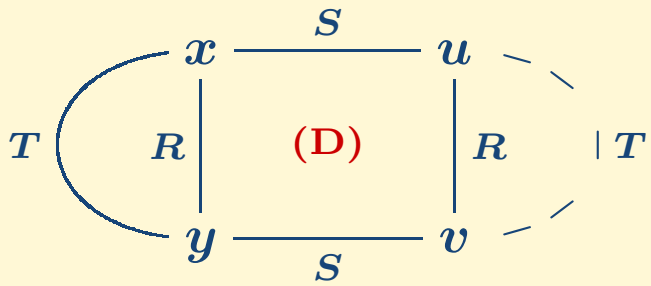

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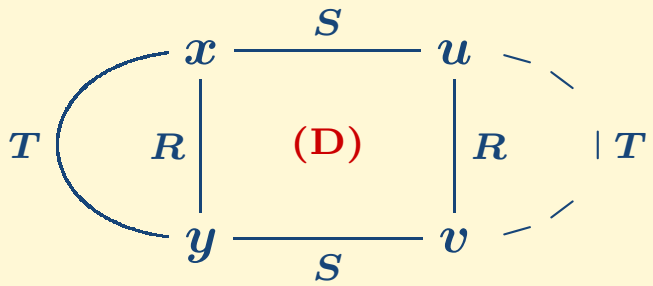

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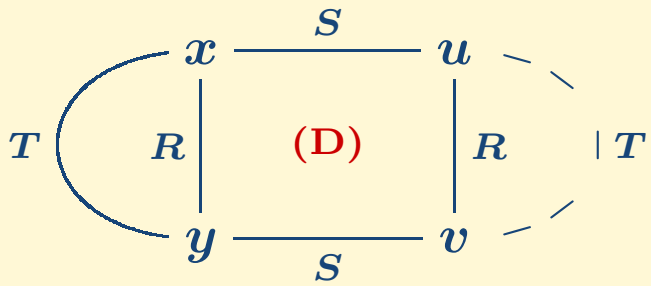

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- Obs: $n \geqslant 4$, n-permutability $\not\Rightarrow$ SL

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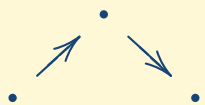

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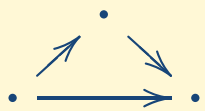

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References

Bourn, **Malcev categories and fibration of pointed objects**, ACS (1996)
Bourn, **The denormalized 3×3 lemma**, JPPA (2003)
Bourn, Gran, **Centrality and connectors in Maltsev categories**, Algebra Universalis (2002)
Bourn, Gran, **Categorical Aspects of Modularity**, Fields Instit. Commun. (2004)
Bourn, Gran, **Normal sections and direct product decompositions**, Comm. Algebra. (2004)
Carboni, Kelly, Pedicchio, **Some remarks on Maltsev and Goursat categories**, ACS (1993)
Carboni, Lambek, Pedicchio, **Diagram chasing in Mal'cev categories**, JPAA (1991)
Carboni, Pedicchio, Pirovano, **Internal graphs and internal groupoids in Mal'cev categories** (1992)
Gran, Rodelo, **A new characterisation of Goursat categories**, ACS (2012)
Gran, Rodelo, **The Cuboid Lemma and Mal'tsev categories**, ACS (2014)
Gran, Rodelo, **Beck-Chevalley condition and Goursat categories,**, JPAA (2017)
Gran, Rodelo, Nguefeu, **Some Remarks on Connectors and Groupoids in Goursat cats**, LMCS (2017)
Gran, Rodelo, Nguefeu, **n-permutable categories and positive relations**, under construction.
Gumm, **Geometrical methods in congruence modular algebras**, Mem. Amer. Math. Soc. (1983)
Hagemann, Herrmann, **A concrete ideal multiplication for algebraic ...**, Algebra Universalis (1979)
Janelidze, **A history of selected topics in categorical algebra I: From Galois theory ...** CGASA (2016)
Janelidze, Pedicchio, **Pseudogroupoids and commutators**, TAC (2001)
Kiss, **Three remarks on the modular commutator**, Algebra Universalis (1992)
Lack, **The 3-by-3 lemma for regular Goursat categories** Homology, Homotopy, and Appl. (2004)
Pedicchio, **A categorical approach to commutator theory**, J. Algebra 177 (1995)
Smith, **Mal'cev varieties**, Lecture Notes in Math., vol. 554, Springer, 1976
Tull, **Conditions for an n-permutable category to be Mal'tsev**, CTGD (2017)