A topos-theoretic approach to systems and behavior

David I. Spivak* and Patrick Schultz

Mathematics Department
Massachusetts Institute of Technology

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Outline

An example system

The National Airspace System (NAS)

- Safe separation problem:
    - Planes need to remain at a safe distance.
    - Can't generally communicate directly.
    - Use radars, pilots, ground control, radios, and TCAS.[1]

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- Systems of systems:
  - A great variety of interconnected systems.
  - Work in concert to enforce global property: safe separation.

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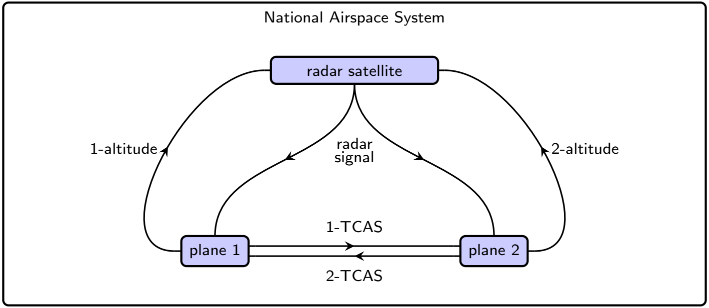
Systems of interacting systems in the NAS



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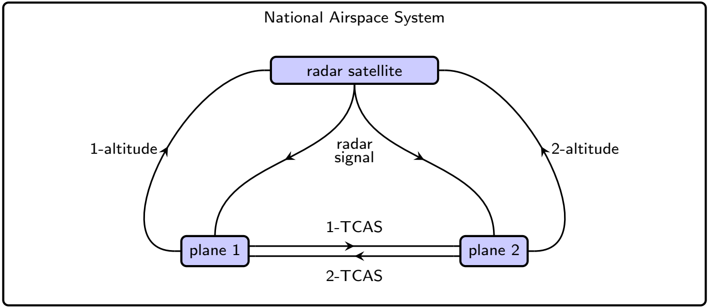
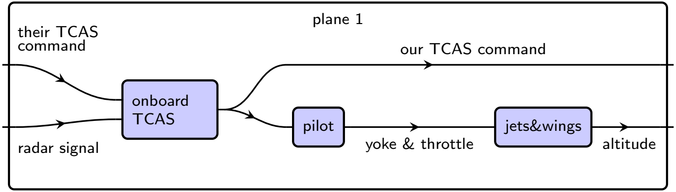
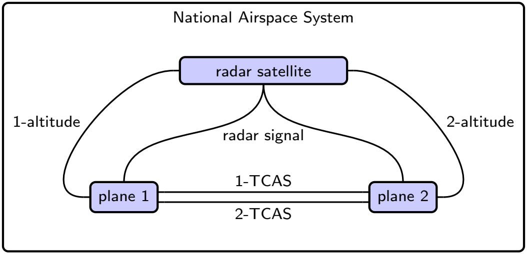




Behavior contracts as predicates



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- ... wire: a sheaf.
- ... box: a predicate—a behavior contract—on the product of its wires.

Prove that if each box's predicate is satisfied, safe separation is achieved.

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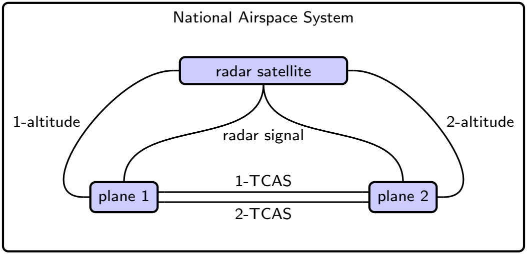

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We'll discuss such a situation using topos theory.

NAS use-case as guide

What's the topos for the National Airspace System?

- This question was a major guide for our work.
- Need to combine many common frameworks into a "big tent".
  - Differential equations, continuous dynamical systems.
  - Labeled transition systems, discrete dynamical systems.
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Relationship to toposes:

- Toposes have an associated internal language and logic.
- Can use formal methods (proof assistants) to prove properties of NAS.

Plan of the talk

1. Define a topos $\mathcal{B}$ of behavior types.

2. Discuss *temporal type theory*, which is sound in $\mathcal{B}$.

3. Return to a NAS use-case.

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We want to model various types of behavior.

- What is a behavior type?
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  - Both are collections of possibilities, indexed by time intervals.
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So what should we mean by time?

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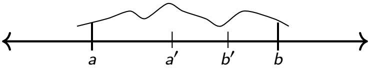
    - "Composition": $B(a, b) = B(a, b') \times_{B(a',b')} B(a', b)$.



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Two reasons *not to use* $\mathrm{Shv}(\mathbb{R})$ as our topos.

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Solution:

- Replace $\mathbb{R}$ with an intervallic timeline, and...
- ... quotient by translation action.

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  - Its poset of points determines a topology...
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This space, $\mathbb{IR}$ is our timeline, and its points are intervals.

$\mathrm{Shv}(\mathbb{IR})$: behaviors in the context of time

Each $X \in \mathrm{Shv}(\mathbb{IR})$ is a behavior type occurring *in the context of time*.

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Next up: keep durations, drop the fixed timeline.

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  - (Left-exact comonads are what define quotient toposes.)
  - For $X \in \mathrm{Shv}(\mathbb{IR})$, define $TX \in \mathrm{Shv}(\mathbb{IR})$ by

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  - There is an inhabited object, which we call $\mathtt{Time} \in \mathcal{B}$,
  - And an equivalence $\mathrm{Shv}(\mathbb{IR}) \cong \mathcal{B}/\mathtt{Time}$.
  - Makes precise "$\mathrm{Shv}(\mathbb{IR})$ is behavior types in the context of time."

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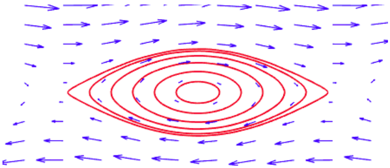

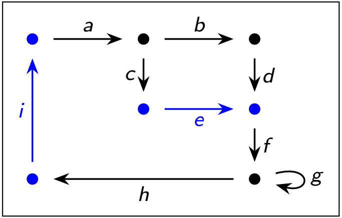

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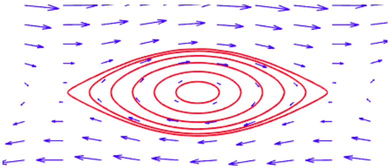

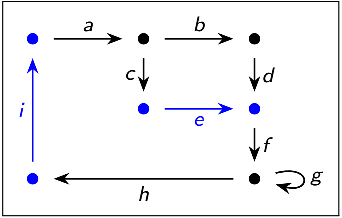

Next up: want logic to define other interesting behaviors.

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Preview of higher-order temporal logic for behavior

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- Logical expressions like the above can be interpreted in the topos $\mathcal{B}$.
- E.g. the above defines a map $P \colon X \to \Omega$, given $B \colon X \to \Omega$.
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Next: use logic to define real "numbers".

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- We can define the type $\underline{\mathbb{R}}$ of *lower reals* internally:

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- Dually, define $\bar{\mathbb{R}}$, with $[\![\, \bar{\mathbb{R}} \,]\!](U) = \{\text{upper semi-continuous} \ldots\}$
- $\underline{\bar{\mathbb{R}}} := \underline{\mathbb{R}} \times \bar{\mathbb{R}}$: *extended intervals*.
- $\mathbb{R} := \{(L, R) : \underline{\bar{\mathbb{R}}} \mid \forall q.\, \neg(Lq \wedge Rq) \wedge \forall (q < q').\, Lq \vee Rq'\}$.

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We refer to $\underline{\mathbb{R}}, \overline{\mathbb{R}}, \overline{\underline{\mathbb{R}}}, \mathbb{R}$, etc. as *Dedekind numeric objects*.

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TTT axiomatics: find finitely many axioms with which to "do real work".

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Temporal type theory

TTT is a finitely presented sub-language of $\mathcal{B}$'s internal language:

- One atomic predicate symbol, $\mathrm{unit\_speed}\colon \bar{\underline{\mathbb{R}}} \to \Omega$.
  - From here, define $\mathtt{Time} := \{\, t : \bar{\underline{\mathbb{R}}} \mid \mathrm{unit\_speed}(t) \,\}$.
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- All are sound in $\mathcal{B}$
  - We already had $\mathtt{Time} \in \mathcal{B}$ externally in the éntendue $\mathcal{B}$.
  - Check that with that interpretation, the ten axioms hold.

Modalities, @ and π

There are a number of useful modalities (Lawvere-Tierney topologies).

- Modalities are internal monads $j\colon \Omega \to \Omega$ on the subobject classifier.
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- Example 1,2: in the context of $t : \mathtt{Time}$, have modalities $\downarrow^t_{[a,b]}, @^t_{[a,b]}$.
  - $\downarrow^t_{[a,b]} P := P \vee (a < t \vee t < b)$.
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We can use these modalities to define *local Dedekind numeric types*.

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  - When $j = \mathrm{id}$ this is lower semicontinuous fns on $\mathbb{IR}$.
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Now we are equipped to define derivatives.

Derivatives of continuous reals

We can define derivatives internally.

- Semantics of $x : \mathbb{R}_\pi$ is: a continuous function on $\mathbb{R}$.
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- Theorem: $\dot{x}$ externally has semantics of derivative of x.

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  is just a formula in the logic.
- We also define “labeled transition systems” internally...
  - ...given two constant sheaves and two maps $E \rightrightarrows V$.
  - Can more generally define any “hybrid system”.

Outline

Setup of safety problem

Variables to be used, and their types:

$$t : \texttt{Time}. \quad T, P : \mathsf{Cmnd}. \quad a : \mathbb{R}_\pi. \quad \texttt{safe}, \texttt{margin}, \texttt{del}, \texttt{rate} : \mathbb{Q}.$$

What these mean:

$t : \texttt{Time}$.	time-line	(a clock).
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Can prove safe separation

$$\forall (t : \mathtt{Time}). \downarrow_0^t \left(t > \mathtt{del} + \frac{\mathtt{safe}}{\mathtt{rate}} \Rightarrow a \geq \mathtt{safe}\right).$$

Outline

If you're interested in reading more

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