Weak functors for degenerate Trimble 3-categories

Eugenia Cheng

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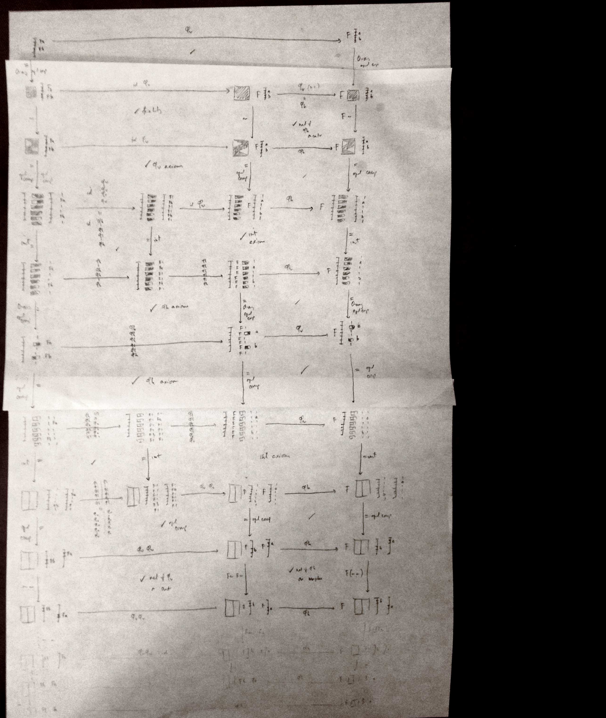

Trimble
n-categories

Trimble
***n*-categories**

doubly degenerate
2-categories

↓

doubly degenerate
3-categories

Trimble
n-**categories**

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Eckmann–Hilton

↓

weak
Eckmann–Hilton

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Eckmann–Hilton

↓

weak
Eckmann–Hilton

distributive
laws

algebras
and maps

↓

strict algebras
and weak maps

Plan

1. Overview

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2. Algebras via distributive laws

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3. Eckmann–Hilton

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3. Eckmann–Hilton
4. Weak Eckmann–Hilton

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4. Weak Eckmann–Hilton
5. Weak maps of algebras.

Test principle for weak n-categories

Doubly degenerate 3-categories should be somehow equivalent to braided monoidal categories.

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Trimble's definition is most like this

Aim: make a 2-category **ddTr3Cat** with

- 0-cells: doubly degenerate Trimble 3-categories
- 1-cells: weak maps
- 2-cells: icon-like transformations (Lack)

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Question: what are weak maps?

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1-cells

2-cells

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$$\left.\begin{array}{l}\text{0-cells}\\[1em]\text{1-cells}\end{array}\right\}\ \text{trivial}$$

2-cells

Warm-up: doubly degenerate 2-categories

0-cells	trivial
1-cells	trivial
2-cells	a set

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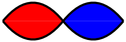

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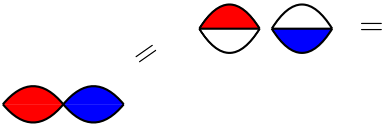

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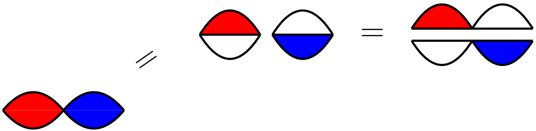

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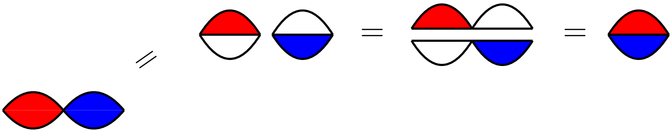

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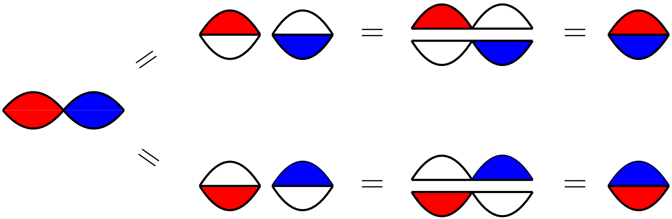

Aim: express this in terms of the monads and algebras

2. Algebras via distributive laws

Let S and T be monads on a category $\mathcal{C}$. (Beck)

A **distributive law** of S over T is a natural transformation

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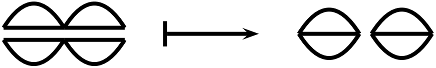

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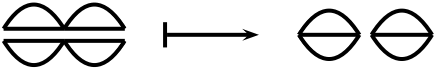

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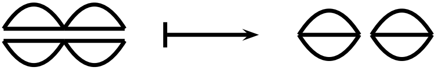

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λ ensures interchange.

Law vs. structure

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For 2-categories this says

> a 2-globular set with vertical and horizontal composition
> compatible via interchange

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$$\Big\downarrow \quad \text{Eckmann–Hilton structure}$$

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T-algebra part is reconstructed

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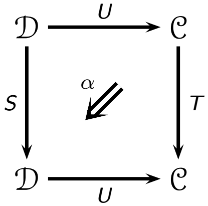

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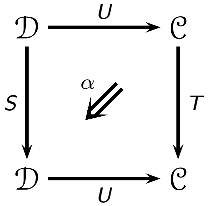
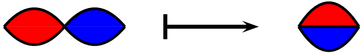
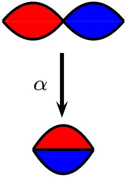


$$TUA \xrightarrow{\alpha} USA$$



Key to Eckmann–Hilton: in a doubly degenerate TS-algebra



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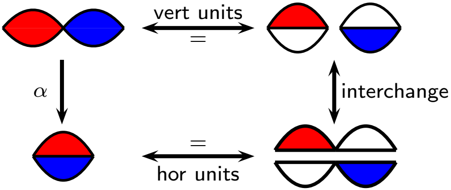

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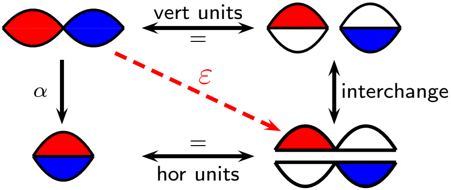

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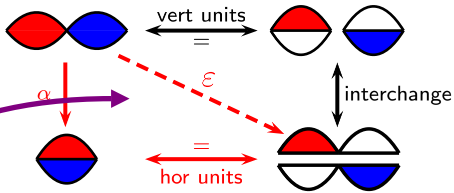

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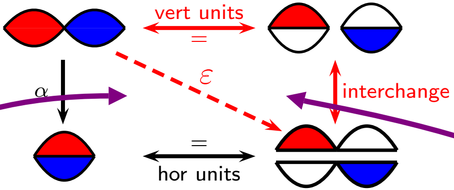

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$$TUA \xrightarrow{\varepsilon_A} STUA \xrightarrow{St} SUA, \qquad TUA \xrightarrow{\alpha_A} SUA$$

and for any S-algebra s

$$TUA \xrightarrow{\varepsilon_A} STUA \xrightarrow{\lambda_A} TSUA \xrightarrow{Ts} TUA, \qquad TUA \xrightarrow{1} TUA$$

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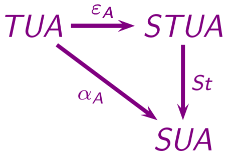

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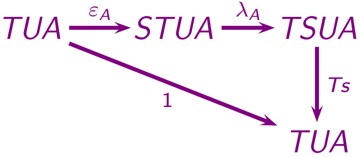

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the following triangle commutes

$$\begin{array}{ccc} TUA & \xrightarrow{\alpha_A} & SUA \\ & {}_t\searrow \quad \swarrow_s & \\ & UA & \end{array}$$

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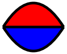

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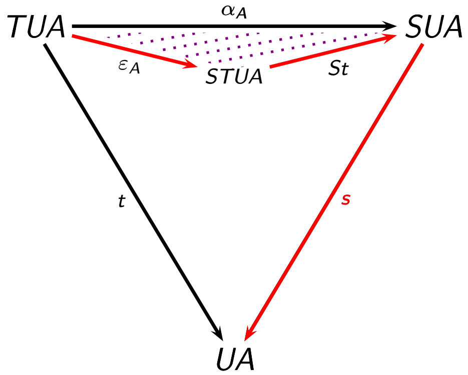

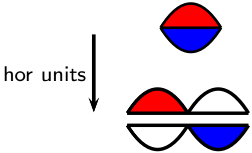

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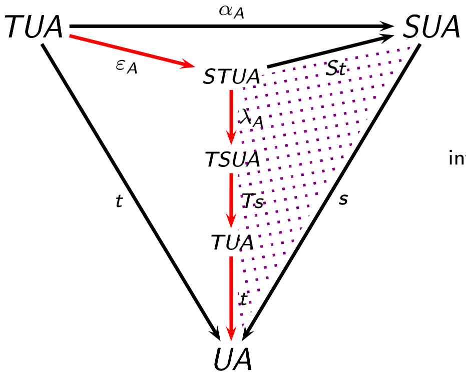

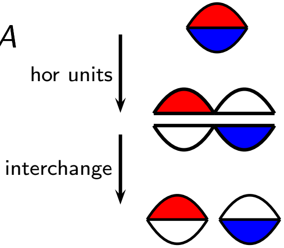

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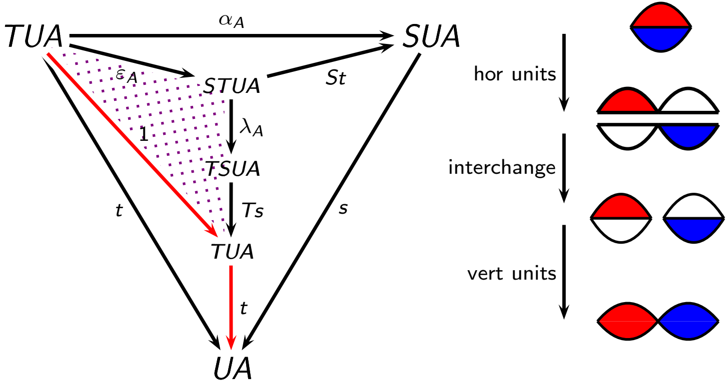

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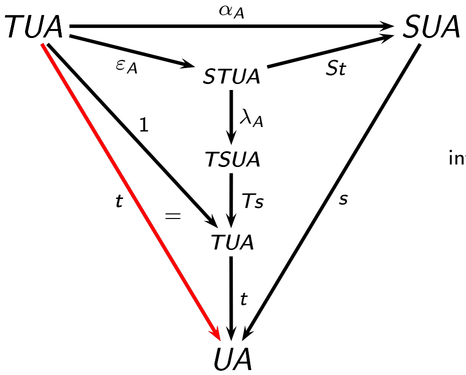

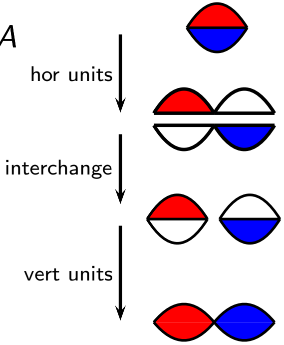

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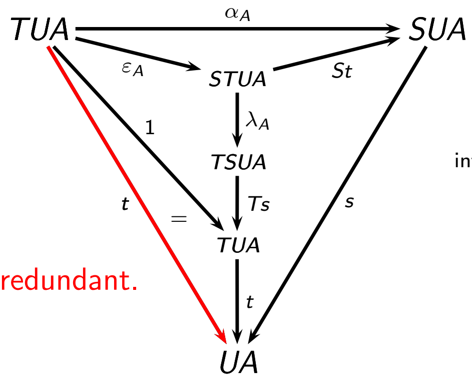

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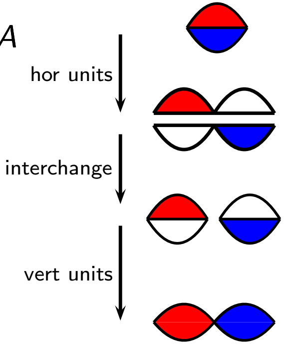

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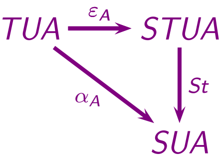

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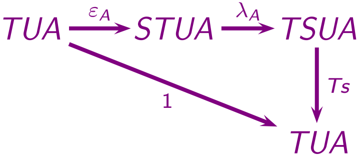

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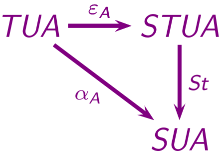

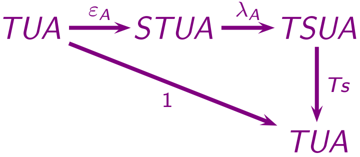

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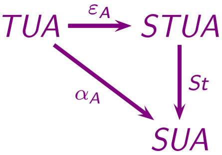

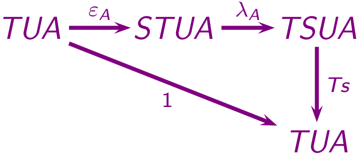

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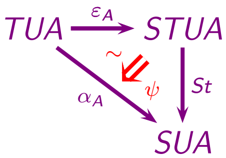

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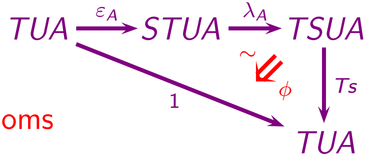

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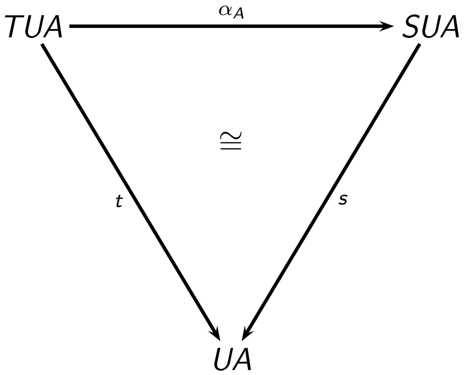

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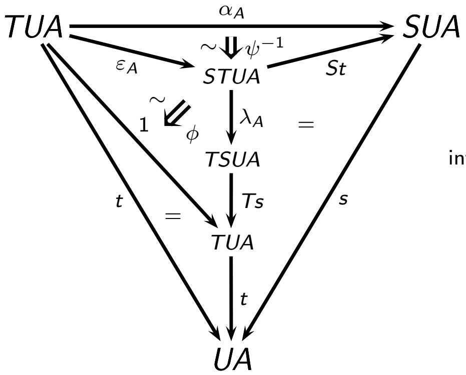

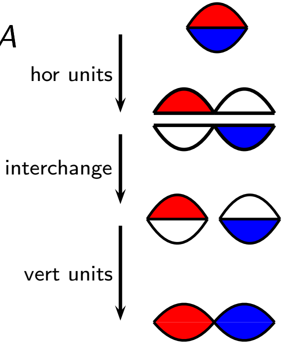

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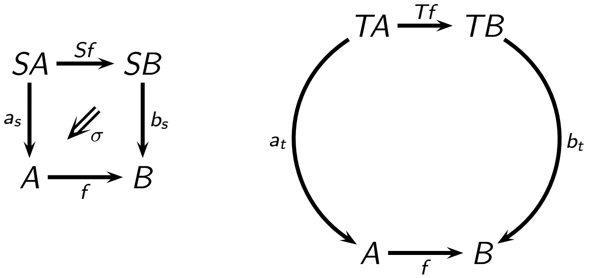

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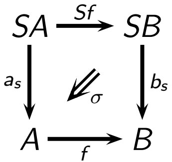

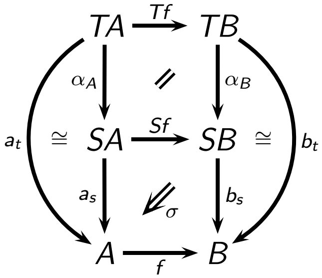

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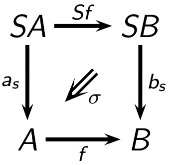

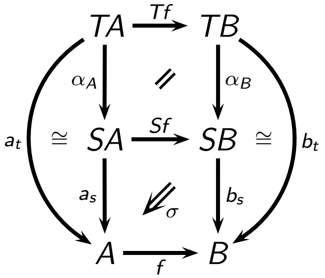

The T-functoriality constraint
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We look at the category of strict TS-algebras and strict maps:

$$TS\text{-}\textbf{Alg} \cong \textbf{Tr3Cat}$$

Weak 3-categories
strict functors.

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This corresponds to
the condition on a monoidal functor making it braided.

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and we get a biequivalence

$$\mathbf{ddTr3Cat} \simeq \mathbf{BrMonCat}$$

The proof follows the methodology of Joyal–Kock.

Abstract E–H: avoid fiddling around with reparametrisations.

Future work (with Nick Gurski)

- Express this at the level of operads and relate it to the little n-cubes operad.
- Examine dependence on weakness of horizontal units, vertical units and distributive law separately.
- Explore using lax duoidal structures. (Batanin–Cisinki–Weber, Garner–López Franco)
- Investigate what type of monads work. (Kelly)
- Better abstract description.
- Relationship between different Eckmann–Hilton structures on the same data.
- Braiding vs. symmetry