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A proof of the model-independence of $(\infty, 1)$-category theory

joint with Dominic Verity

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Plan

Goal: build model-independent foundations of $(\infty, 1)$-category theory

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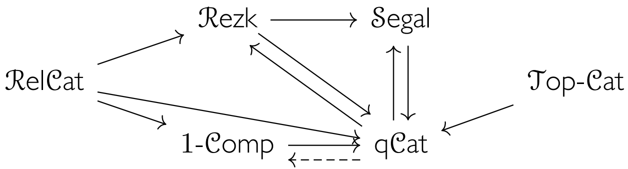
1. What are model-independent foundations?
2. ∞-cosmoi of $(\infty, 1)$-categories
3. A taste of the formal category theory of $(\infty, 1)$-categories
4. The proof of model-independence of $(\infty, 1)$-category theory

I

What are model-independent foundations?

Models of $(\infty, 1)$-categories

Schematically, an $(\infty, 1)$-category is a category "weakly enriched" over ∞-groupoids/homotopy types ... but this is tricky to make precise.



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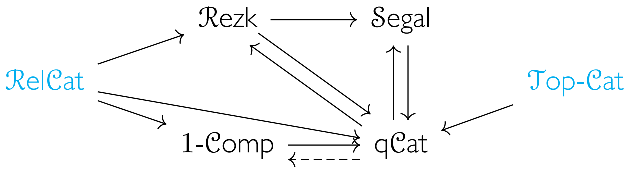

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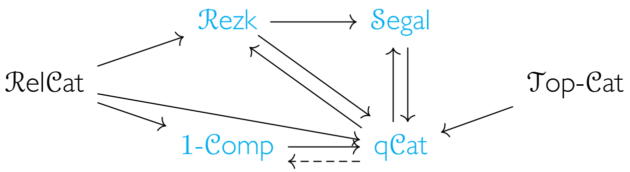

- topological categories and relative categories are the simplest to define but do not have enough maps between them
- $\left\{\begin{array}{l}\text{quasi-categories (nee. weak Kan complexes),}\\ \text{Rezk spaces (nee. complete Segal spaces),}\\ \text{Segal categories, and}\\ \text{(saturated 1-trivial weak) 1-complicial sets}\end{array}\right.$
  each have a homotopically meaningful internal hom.

The analytic vs synthetic theory of $(\infty, 1)$-categories

Q: How might you develop the category theory of $(\infty, 1)$-categories?

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Two strategies:

- work analytically to give categorical definitions and prove theorems using the combinatorics of one model

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Our method: introduce an ∞-cosmos to axiomatize common features of the categories $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ of $(\infty, 1)$-categories.

2

∞-cosmoi of $(\infty, 1)$-categories

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Henceforth ∞-category and ∞-functor are technical terms that mean the objects and morphisms of some ∞-cosmos.

The homotopy 2-category

The homotopy 2-category of an ∞-cosmos is a strict 2-category whose:

- objects are the ∞-categories A, B in the ∞-cosmos
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Prop. Equivalences in the homotopy 2-category

$$A \underset{g}{\overset{f}{\rightleftarrows}} B \qquad A \underset{gf}{\overset{1_A}{\underset{\Downarrow\cong}{\rightrightarrows}}} A \qquad B \underset{fg}{\overset{1_B}{\underset{\Downarrow\cong}{\rightrightarrows}}} B$$

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Thus, non-evil 2-categorical definitions are "homotopically correct."

3

A taste of the formal category theory of $(\infty, 1)$-categories

Absolute lifting diagrams

$$\begin{array}{ccc} & & B \\ & \overset{r}{\nearrow} \; \Downarrow\rho & \downarrow f \\ C & \xrightarrow[g]{} & A \end{array}$$

is an absolute right lifting diagram if it and any restriction

are right liftings:

$$\begin{array}{ccc} X & \xrightarrow{b} & B \\ c\downarrow & \forall \Downarrow \chi & \downarrow f \\ C & \xrightarrow[g]{} & A \end{array} \quad = \quad \begin{array}{ccc} X & \xrightarrow{b} & B \\ c\downarrow & \exists ! \Downarrow \zeta \;\; \overset{r}{\nearrow} \;\; \Downarrow \rho & \downarrow f \\ C & \xrightarrow[g]{} & A \end{array},$$

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- $$\begin{array}{ccc} & & E \\ & \overset{s}{\nearrow}\ \Downarrow\sigma & \downarrow k \\ C & \xrightarrow[r]{} & B \end{array}$$ is absolute right lifting iff $$\begin{array}{ccc} & & E \\ & \overset{s}{\nearrow}\ \Downarrow\sigma & \downarrow k \\ & & B \\ & \overset{r}{\nearrow}\ \Downarrow\rho & \downarrow f \\ C & \xrightarrow[g]{} & A \end{array}$$ is

Adjunctions and limits

An adjunction between ∞-categories is an adjunction $(A, B, f, u, \eta, \epsilon)$ in the homotopy 2-category.

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⇝ Hence all 2-categorical theorems about adjunctions become theorems about adjunctions between ∞-categories! In particular:

A right adjoint $B \underset{u}{\overset{f}{\underset{\perp}{\rightleftarrows}}} A$ is an absolute right lifting

$$\begin{array}{ccc} & & B \\ & \overset{u}{\nearrow} \;\Downarrow\epsilon & \downarrow f \\ A & = & A \end{array}$$

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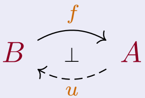
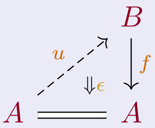
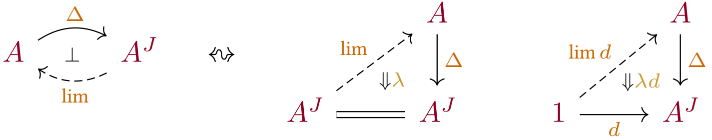
A right adjoint 

is an absolute right lifting 

Hence, a limit functor or limit of $d \colon 1 \to A^J$ is an absolute right lifting



Right adjoints preserve limits

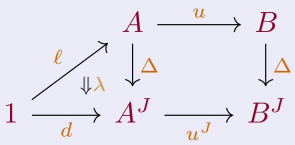
Prop (right adjoints preserve limits). If $A \underset{u}{\overset{f}{\underset{\bot}{\rightleftarrows}}} B$ and $\lambda\colon \Delta\ell \Rightarrow d$ is a limit cone then

$$\begin{array}{ccccc} & & A & \xrightarrow{u} & B \\ & \overset{\ell}{\nearrow} & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} \\ 1 & \underset{d}{\longrightarrow} & A^J & \underset{u^J}{\longrightarrow} & B^J \end{array} \quad (\Downarrow\lambda)$$

is absolute right lifting.

Right adjoints preserve limits

Prop (right adjoints preserve limits). If $A \underset{u}{\overset{f}{\leftrightarrows}} B$ with $f \dashv u$ and $\lambda \colon \Delta\ell \Rightarrow d$ is a limit cone then



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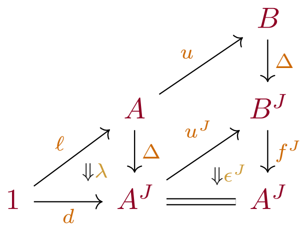
Proof: It suffices to show the transposed cone is absolute right lifting



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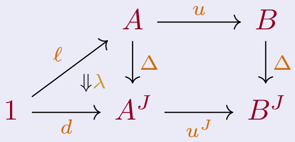

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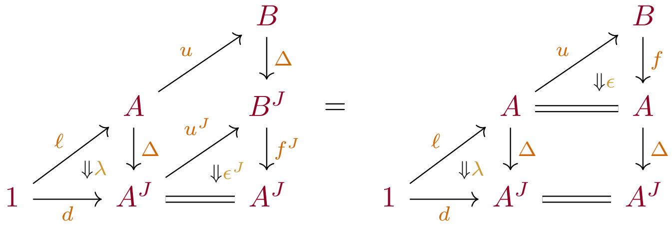

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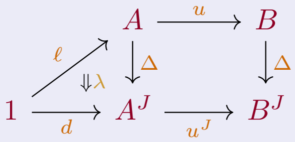

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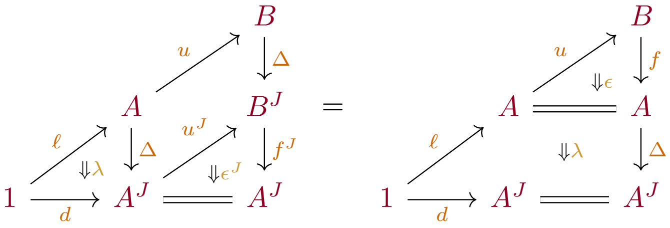

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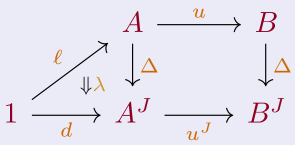

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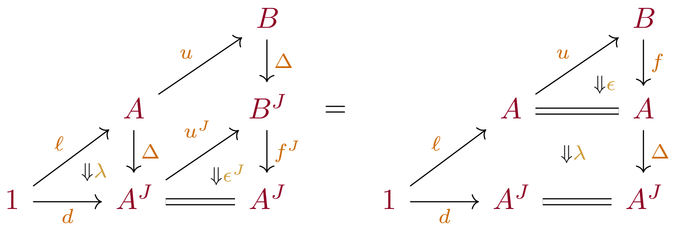

which is the case by 2-naturality and composition of absolute right liftings.

Universal properties of adjunctions and limits

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Any cospan has a comma ∞-category

$$\begin{array}{ccccc} & & \mathrm{Hom}_A(f,g) & & \\ & \overset{\mathrm{cod}}{\swarrow} & & \overset{\mathrm{dom}}{\searrow} & \\ C & & \overset{\phi}{\Leftarrow} & & B \\ & \underset{g}{\searrow} & & \underset{f}{\swarrow} & \\ & & A & & \end{array}$$

with

comma span a two-sided discrete fibration aka a module $C \overset{\mathrm{Hom}_A(f,g)}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} B$.

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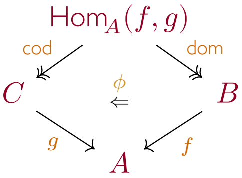

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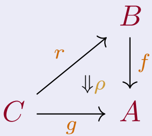

absolute lifting iff $\mathrm{Hom}_B(B,r) \underset{C\times B}{\simeq} \mathrm{Hom}_A(f,g)$.

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Cor. $A \underset{u}{\overset{f}{\rightleftarrows}} B$ with $f \dashv u$ iff $\mathrm{Hom}_A(f,A) \simeq_{A\times B} \mathrm{Hom}_B(B,u)$.

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Cor. $A \underset{u}{\overset{f}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} B$ iff $\mathrm{Hom}_A(f,A) \simeq_{A\times B} \mathrm{Hom}_B(B,u)$.

Cor. $d\colon 1 \to A^J$ has a limit ℓ iff $\mathrm{Hom}_A(A,\ell) \simeq_A \mathrm{Hom}_{A^J}(\Delta, d)$.

The calculus of modules

Thm. Any ∞-cosmos has a virtual equipment of ∞-categories, ∞-functors, modules, and “multilinear” module maps:

$$\begin{array}{ccccccc}
A_0 & \overset{E_1}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & A_1 & \overset{E_2}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & \cdots & \overset{E_n}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & A_n \\
{\scriptstyle f}\downarrow & & & \Downarrow\alpha & & & \downarrow{\scriptstyle g} \\
B_0 & & & \underset{F}{\xrightarrow{\hspace{4em}+\hspace{4em}}} & & & B_n
\end{array}$$

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with units $\begin{array}{ccc} A & = & A \\ \| & \Downarrow\iota & \| \\ A & \xrightarrow[\mathrm{Hom}_A]{} & A \end{array}$ and restriction of scalars $\begin{array}{ccc} X & \xrightarrow{E(b,a)} & Y \\ {\scriptstyle a}\downarrow & \Downarrow\rho & \downarrow{\scriptstyle b} \\ A & \xrightarrow[E]{} & B \end{array}$

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$\rightsquigarrow$ The homotopy 2-category embeds covariantly and contravariantly.

The calculus of modules

Thm. Any ∞-cosmos has a virtual equipment of ∞-categories, ∞-functors, modules, and "multilinear" module maps:

$$\begin{array}{ccccccc} A_0 & \overset{E_1}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\;} & A_1 & \overset{E_2}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\;} & \cdots & \overset{E_n}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\;} & A_n \\ {\scriptstyle f}\downarrow & & & \Downarrow\alpha & & & \downarrow{\scriptstyle g} \\ B_0 & & & \underset{F}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\;} & & & B_n \end{array}$$

with units
$$\begin{array}{ccc} A & =\!\!\!+\!\!\!= & A \\ \| & \Downarrow\iota & \| \\ A & \underset{\mathrm{Hom}_A}{\dashrightarrow} & A \end{array}$$
and restriction of scalars
$$\begin{array}{ccc} X & \overset{E(b,a)}{\dashrightarrow} & Y \\ {\scriptstyle a}\downarrow & \Downarrow\rho & \downarrow{\scriptstyle b} \\ A & \underset{E}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\;} & B \end{array}$$

$\rightsquigarrow$ The homotopy 2-category embeds covariantly and contravariantly.

Modules $A \overset{E}{\nrightarrow} B$ and $A \overset{F}{\nrightarrow} B$ are isomorphic iff $E \simeq_{A\times B} F$ so the virtual equipment captures the formal category theory of ∞-categories.

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The proof of model-independence of $(\infty, 1)$-category theory

Cosmological biequivalences and change-of-model

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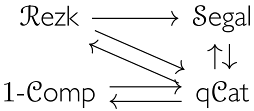
Idea: $FA \simeq A', FB \simeq B' \rightsquigarrow \mathcal{K}_{/A\times B} \xrightarrow{\sim} \mathcal{L}_{/FA\times FB} \xrightarrow{\sim} \mathcal{L}_{/A'\times B'}$

Model-independence



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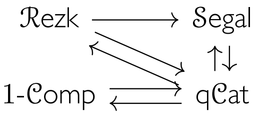

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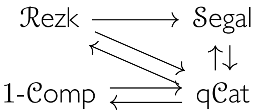

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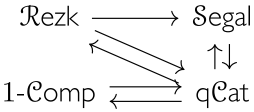

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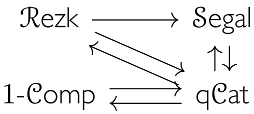

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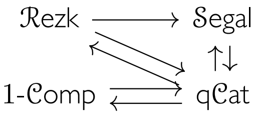

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Analytically-proven theorems also transfer along biequivalences:

- Universal properties in an $(\infty, 1)$-category A are determined elementwise, by each $a \colon 1 \to A$.

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- Both analytically- and synthetically-proven results about $(\infty, 1)$-categories transfer across "change-of-model" functors called biequivalences.
- Open problems: many $(\infty, 1)$-categorical notions are yet to be incorporated into ∞-cosmology.

References

For more on the model-independent theory of $(\infty, 1)$-categories see:

Emily Riehl and Dominic Verity

- mini-course lecture notes:

  ∞-Category Theory from Scratch
  arXiv:1608.05314

- draft book in progress:

  Elements of ∞-Category Theory
  www.math.jhu.edu/~eriehl/elements.pdf

Obrigada!