Affine objects in a tangent category

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Tangent categories

- A category $\mathbb{X}$ equipped with an endofunctor $T : \mathbb{X} \longrightarrow \mathbb{X}$ with various additional structure which the tangent bundle functor on the category of smooth manifolds satisfies.
- First defined by Rosický in 1984: wanted to find a common abstraction of categories in synthetic differential geometry (SDG) and the category of smooth manifolds.
- Rediscovered by Cockett and Cruttwell in 2013: wanted to find a common abstraction for differential categories and the category of smooth manifolds.
- Since then, progress on two different fronts: additional models for the axioms and additional theory of tangent categories.

Models

- Smooth manifolds.
- Convenient vector spaces and convenient manifolds.
- C^{∞}-rings, models of SDG.
- Commutative rings, affine schemes, schemes.
- Differential lambda calculus.
- **Abelian functor calculus (Bauer et. al. 2017).**
- **Further possibilities for functor calculus (Work-in-progress of Bauer, Burke, Ching, based on ideas of Goodwillie).**

Theory

Many concepts from differential geometry can be defined in tangent categories:

- Vector fields and their Lie bracket.
- Differential and sector forms.
- Vector bundles.
- Connections.
- Differential equations and their solutions.
- **Symplectic geometry, Noether's theorem (MacAdam 2018)**

Most of these definitions require significant modifications to fit into the axiomatics!

Tangent category definition

Definition (Rosický 1984, modified Cockett/Cruttwell 2013)

A **tangent category** consists of a category $\mathbb{X}$ with:

- **tangent bundle functor**: an endofunctor $T : \mathbb{X} \longrightarrow \mathbb{X}$;
- **projection of tangent vectors**: a natural transformation $p : T \longrightarrow 1_{\mathbb{X}}$;
- for each M, the pullback of n copies of p_M along itself exists (and is preserved by each T^m), call this pullback $T_n M$;
- **addition and zero tangent vectors**: for each $M \in \mathbb{X}$, p_M has the structure of a commutative monoid in the slice category $\mathbb{X}/M$; in particular there are natural transformations $+ : T_2 \longrightarrow T$, $0 : 1_{\mathbb{X}} \longrightarrow T$;

Tangent category definition (continued)

Definition

- **symmetry of mixed partial derivatives**: a natural transformation $c : T^2 \longrightarrow T^2$;
- **linearity of the derivative**: a natural transformation $\ell : T \longrightarrow T^2$;
- **the vertical bundle of the tangent bundle is trivial**:

$$\begin{tikzcd}
T_2(M) \arrow[rr, "\langle \pi_0 \ell, \pi_1 0_{TM} \rangle T(+)"] \arrow[d, "\pi_0 p_M = \pi_1 p_M"'] & & T^2(M) \arrow[d, "T(p_M)"] \\
M \arrow[rr, "0_M"'] & & T(M)
\end{tikzcd}$$

  is a pullback;
- various coherence equations for ℓ and c.

Note: Building on work of Leung, Garner has shown how tangent categories are a type of enriched category.

Overview

- Today: how to generalize "affine manifolds" to tangent categories.
- Affine manifolds: a choice of chart such that each transition map is affine.
- By a result of Auslander and Markus in 1955, can be equivalently described as a smooth manifold equipped with a flat, torsion-free connection.
- We use this as the definition in a tangent category.

"Old theorems don't die, they just become definitions".

Differential objects

Differential objects are the simplest kind of object in a tangent category: they are objects for which the tangent bundle splits nicely.

Definition

A **differential object** is a commutative monoid M together with a map $\hat{p} : TM \longrightarrow M$ satisfying various equational axioms, and such that

$$\begin{tikzcd} & TM \arrow[dl, "\hat{p}"'] \arrow[dr, "p_M"] & \\ M & & M \end{tikzcd}$$

is a product diagram.

Examples: Cartesian spaces $\mathbb{R}^n$ in smooth manifolds, KL-vector spaces in SDG, polynomial rings in $\mathrm{cRing}^{\mathrm{op}}$ (should have: stable categories in functor calculus).

Connections

Connections are a splitting of the second tangent bundle.

Definition

(Cockett/Cruttwell 2016, Lucyshyn-Wright 2017) A **connection** on (the tangent bundle of) M consists of a map $K : T^2M \longrightarrow TM$ satisfying various equational axioms, and such that

$$\begin{tikzcd} & T^2M \arrow[dl, "K"'] \arrow[d, "p_{TM}"] \arrow[dr, "T(p_m)"] & \\ TM \arrow[dr, "p_M"'] & TM \arrow[d, "p_M"] & TM \arrow[dl, "p_M"] \\ & M & \end{tikzcd}$$

is a limit diagram.

For smooth manifolds, this is equivalent to the covariant derivative definition.

Flat and torsion-free

Definition

A connection $K : T^2M \longrightarrow TM$ is called

- **torsion-free** if:

$$\begin{tikzcd} T^2M \arrow[r, "c_M"] \arrow[dr, "K"'] & T^2M \arrow[d, "K"] \\ & TM \end{tikzcd}$$

- **flat** if

$$\begin{tikzcd} T^3M \arrow[r, "c_{TM}"] \arrow[d, "T(K)"'] & T^3M \arrow[r, "T(K)"] & T^2M \arrow[d, "K"] \\ T^2M \arrow[rr, "K"'] & & TM \end{tikzcd}$$

An **affine object** is an object M equipped with a flat, torsion-free connection K.

For smooth manifolds, these are equivalent to the usual definitions.

Differential and affine objects

(Differential) $\subset$ (Affine) $\subset$ (Object with connection):

- Differential objects have a canonical affine structure.
- Affine objects are more general: eg., in smooth manifolds, the torus, Moebius band and Klein bottle can all be given affine structure.
- All smooth manifolds have a torsion-free connection, but not all have a *flat* torsion-free connection: eg., no S^n for $n > 1$ admits an affine structure.

Categories of objects with a connection

Definition

- If (M, K) and (M', K') are objects equipped with connections, a **morphism** between them is a map $f : M \longrightarrow M'$ in $\mathbb{X}$ so that

$$\begin{tikzcd} T^2M \arrow[r, "T^2(f)"] \arrow[d, "K"'] & T^2M' \arrow[d, "K'"] \\ TM \arrow[r, "T(f)"'] & T^2M' \end{tikzcd}$$

- Let the **geometric category**, $\mathrm{Geom}(\mathbb{X}, T)$, be the category of such objects and morphisms.
- Let the **affine category**, $\mathrm{Aff}(\mathbb{X}, T)$, be the full subcategory of $\mathrm{Geom}(\mathbb{X}, T)$ consisting of the affine objects.

Interestingly, such morphisms are almost unheard of in differential geometry...

Geometric and affine categories as tangent categories

Theorem

Both Geom$(\mathbb{X}, T)$ *and Aff*$(\mathbb{X}, T)$ *are tangent categories, when equipped with the endofunctor* T^* *defined by*

$$T^*(M, K) := (TM, K_{TM})$$

where K_{TM} *is defined as the composite*

$$T^3M \xrightarrow{T(c_M)} T^3M \xrightarrow{c_{TM}} T^3M \xrightarrow{T(K)} T^2M \xrightarrow{c_M} T^2M$$

and using the same natural transformations as in $(\mathbb{X}, T)$.

Question: what is a connection on an object in these tangent categories?
Specifically: when is K *itself* a connection on $(M, K) \in \text{Geom}(\mathbb{X}, T)$?

Characterization of flat torsion-free connections

Theorem

If $K : T^2M \longrightarrow TM$ is a connection on M, then the following are equivalent:

- *K is flat and torsion-free (that is, (M, K) is affine);*
- *$K \circ K_T = K \circ T(K)$;*
- *K is a map in Geom$(\mathbb{X}, T)$ from $T^*(T^*(M, K))$ to $T^*(M, K)$;*
- *K is a connection on (M, K) in the tangent category Geom$(\mathbb{X}, T)$.*

(As far as we know, this is new in differential geometry).

Theorem

There is a 2-comonad on the 2-category of tangent categories, Aff, which on objects sends $(\mathbb{X}, T)$ to the tangent category Aff$(\mathbb{X}, T)$.

Jubin's thesis

- In a recent thesis, Jubin investigated the algebras for the canonical monad structure on the tangent bundle functor.
- He also showed that the tangent bundle on the category of affine monads had an infinite number of monad and comonad structures on it, as well as distributive laws relating these monads and comonads.
- Here we briefly sketch how to generalize these ideas in the affine category of a tangent category.

Monad structure of the tangent bundle functor

- The tangent bundle functor on smooth manifolds is a monad.
- If we let $\langle x, v\rangle$ denote an element of TM, and $\langle x, v, w, d\rangle$ an element of T^2M, then the multiplication $\mu : T^2M \longrightarrow TM$ is defined by

$$\langle x, v, w, d\rangle \mapsto \langle x, v + w\rangle,$$

while the unit η is simply the 0 vector field: $x \mapsto \langle x, 0\rangle$.
- This works in any tangent category, with $\eta := 0$ and μ defined as the composite

$$T^2M \xrightarrow{\langle T(p_M), p_{TM}\rangle} T_2M \xrightarrow{+} TM.$$

Jubin has shown that algebras of this monad are related to foliations (!).

A different monad on affine objects

- Jubin showed that one could define many different monad structures on the tangent bundle functor on the category of affine manifolds: the first of these is $\mu^1 : T^2M \longrightarrow TM$, defined in local coordinates by mapping
$$\langle x, v, w, d \rangle \mapsto \langle x, v + w + d \rangle.$$
- More generally, for any $a \in \mathbb{R}$, he defines $\mu^a : T^2M \longrightarrow TM$ by
$$\langle x, v, w, d \rangle \mapsto \langle x, v + w + a \cdot d \rangle.$$
- We can define these by making use of the map $K : T^2M \longrightarrow TM$.

Monads on the affine category

Theorem

On the affine category of a tangent category, for any $a \in \mathbb{Z}_{\geq 0}$, there is a monad $(T, 0, \mu^a)$, where $\mu^a_{(M,K)}$ is defined as the composite

$$T^2M \xrightarrow{\langle K, K, \ldots, K, T(p_M), p_{TM} \rangle} T_{a+2}(M) \xrightarrow{+} TM$$

(where there are a copies of K).

Comonads on the affine category

Any connection has an associated "horizontal lift" map

$$H : T_2M \longrightarrow T^2M.$$

This can be used to define comonads on the affine category:

Theorem

On the affine category of a tangent category, for any $a \in \mathbb{Z}_{\geq 0}$, there is a comonad (T, p, δ^a); in particular, $\delta^0_{(M,K)}$ is defined as the composite

$$TM \xrightarrow{\langle 1, 1 \rangle} T_2(M) \xrightarrow{H} T^2M.$$

Distributive laws

The following is new, even for smooth manifolds:

Theorem

$c : T^2 \longrightarrow T^2$ is a distributive law of any of the monads/comonads over any of the monads/comonads.

Additionally, assuming the tangent category has negatives, we can generalize some of Jubins's results:

- for any a, b, there is a bimonad

$$(T, 0, p, \mu^a, \delta^b, \lambda^{a,b})$$

  (with a different distributive law than above);
- for $a = 0$ or $b = 0$, negation $n : TM \longrightarrow TM$ makes this into a Hopf monad.

Conclusions

In conclusion:

- Affine objects can be defined in any tangent category; they are the "next simplest objects" after differential objects.
- The affine objects of a tangent category, with appropriate morphisms, also form a tangent category.
- Thinking about affine objects in these terms leads to new characterizations of flat, torsion-free connections.
- The affine category of a tangent category is very richly structured, with many monads, comonads, and distributive laws.

Future work

Lots more to be done:

- Objects with "higher-order connections".
- Greater understanding of the categories of algebras of these monads and comonads (all the algebra categories are themselves tangent categories).
- **Affine objects in Goodwillie's functor calculus** (eg., Goodwillie has described the category of spaces as having two distinct flat torsion-free connections).

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