An axiomatic approach to Gabriel-Ulmer duality

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Structure of the talk

- Give a definition of accessible and (locally) presentable object in a 2-category.

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- Cast a Gabriel-Ulmer duality for this definition.

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- Y is representably fully faithful + something else.

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Remark

This does not imply that $P(G)$ is accessible.

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Representation Theorem

The following are equivalent:

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Well, maybe one should start by recalling what Gabriel Ulmer duality is.

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Can we cast a Gabriel Ulmer duality for such a weak notion of accessibity? Not really.

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Definition

Given a context $S \overset{Y}{\Rightarrow} P$ a Gabriel Ulmer envelope $\widehat{(\_)}$ for Y is an addition KZ monad such that

$$S(\widehat{(\_)}) \cong P(\_)$$

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Gabriel Ulmer Duality

Let $S \overset{Y}{\Rightarrow} P$ be a Yoneda context and $\widehat{(\_)}$ GU envelope for Y. If

- $\widehat{(\_)}$ is soaking;
- S is climbable;

then

$$\mathrm{Alg}(\widehat{(\_)})^{\mathrm{op}} \cong \mathrm{Pres}(Y).$$

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\mathrm{Ind}(G) \arrow[rr, shift left, "i"] \arrow[rr, phantom, "\top"] & & \mathrm{Ind}(G') \arrow[ll, shift left, "L"]
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then we know that L preserves compact objects, i.e. the dotted arrow exists.

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