Generalized symmetries and arithmetic applications

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- Today: open questions, the work of other people, some of my own

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- Property of existence $\to$ a structure

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A *p-derivation* on R is a function $\delta\colon R \to R$ modeled on

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$\{p\text{-derivations on } R\} \xrightarrow{\sim} \{\text{Frobenius lifts on } R\}$, if R is p-tor-free

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$$\begin{aligned} W^{\mathrm{diff}}(R) &= \Big\{ \sum_n a_n \frac{t^n}{n!} \ \Big|\ a_n \in R \Big\}, \quad d = d/dt \\ &= \{(a_0, a_1, \dots)\}, \quad d = \text{shift} \end{aligned}$$

Multiplication law at the n-th component is given by the Leibniz rule for $d^{\circ n}(xy)$:

$$(a_0, \dots) \times (b_0, \dots) = (a_0 b_0,\ a_0 b_1 + a_1 b_0,\ a_0 b_2 + 2a_1 b_1 + a_2 b_0,\ \dots)$$

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$$W(R) = R \times R \times R \times \cdots, \quad \delta(a_0, a_1, \ldots) = (a_1, a_2, \ldots)$$

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- Better: Witt vectors are a machine for adding a Frobenius lift to your ring, interpreted in an intelligent way

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- Thus, if one is sufficiently enlightened, the concept of Frobenius lift, or p-derivation, leads automatically to crystalline cohomology.

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A composition object of C is an object P of C plus a comonad structure on the functor it represents. ('Tall–Wraith monad object')

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- An element f in a composition ring P is *linear* if it acts additively whenever P acts on a ring

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Generalized symmetries, continued

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  - In fact, the composition ring $\mathbb{Z}[e, \delta, \delta^{\circ 2}, \dots]$ cannot be generated by linear operators! It is fundamentally nonlinear.

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- Is it possible to classify all composition objects in Ring?
  - Carlson: Yes, if we allow denominators
  - Buium: Some positive classification results for composition rings generated by a single operator
  - All known examples come from linear operators or lifting Frobenius-like constructions from char p to char 0.

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- $\mathrm{END}(\mathbb{F}_p[t]) = ?$. Includes derivation d/dt, t-derivation $f \mapsto (f - f^q)/t, \ldots$

III. Generalized-equivariant algebriac geometry

Principal categories of algebraic geometry:

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- Can this be done for *every* composition ring?
- Could there some kind of new generalized symmetry structures that exist only at the non-affine level?

δ-structures on schemes (Greenberg, Buium, me)

Given a functor $X\colon \mathrm{Ring} \to \mathrm{Set}$, define

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- This allows us to extend the theory of p-derivations, δ-structures, and Witt vectors from rings to schemes $\to$ "δ-equivariant algebraic geometry"
- The proof (Illusie, van der Kallen, Langer–Zink, me) is not formal!

Hilbert's 12th Problem

Given a finite extension $K/\mathbb{Q}$, is there an explicit description of K^{ab}, its maximal Galois extension with abelian Galois group?

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- New idea: Use periodic points on Λ_K-schemes instead!

Λ_K-structures

Fix a finite extension $K/\mathbb{Q}$. Let $\mathcal{O}_K$ denote its subring of algebraic integers. Let R be an $\mathcal{O}_K$-algebra.

- A Λ_K-structure on R is a commuting family of endomorphisms $\psi_{\mathfrak{p}}$, one for each nonzero prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ such that $\psi_{\mathfrak{p}}(x) \equiv x^{N(\mathfrak{p})} \mod \mathfrak{p}R$, where $N(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}|$.

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- Wilkerson, Joyal: $\Lambda_{\mathbb{Q}}$-ring $=$ λ-ring as in K-theory

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- Any answer, positive or negative, for any other K would be very interesting!

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  - However, that might be enough in general if there is a classification result for composition rings
- Can we make sense of $\mathrm{END}(X)$ for non-affine schemes?
  - If so, we might hope to find new Λ_K-schemes, and hence say something about Hilbert's 12th Problem, by looking and $\mathrm{END}(X)$ for specific X, say $\mathbb{P}^2_{\mathsf{O}_K}$
- Can one classify the composition objects in $\mathrm{CAlg}_{\mathbb{R}_{\geq 0}}$?
  - There are nonlinear ones! Use positivity instead of integrality!
- There must be many examples of other categories of algebras with generalized symmetries which are interesting and important!