Equivariant fundamental groupoids as categorical constructions

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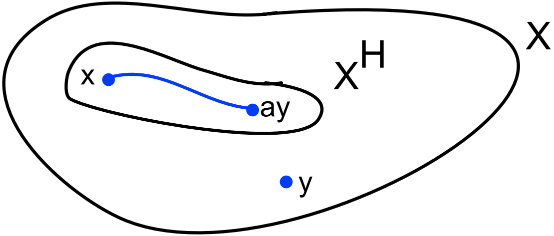
joint with Dorette Pronk

Equivariant Fundamental Groupoid $\Pi_1(G, X)$

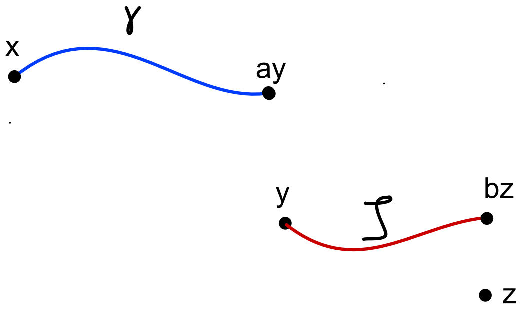
X is a G-space
Define a category (not a groupoid) $\Pi_1(G, X)$ with:

- Objects $(G/H, x)$ with $H \leq G$ and $x \in X^H$
- Arrows: (α, γ) where γ is a path from x to αy in X^H
- This is considered a path from x to y



Composition

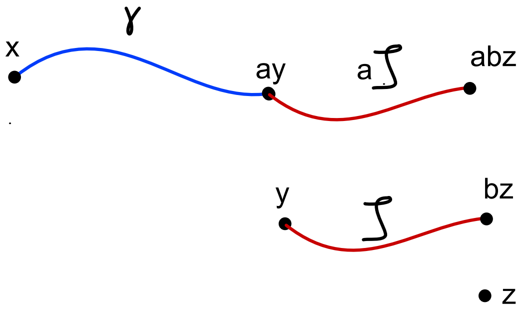
$$(\alpha, \gamma) \circ (\beta, \zeta)$$



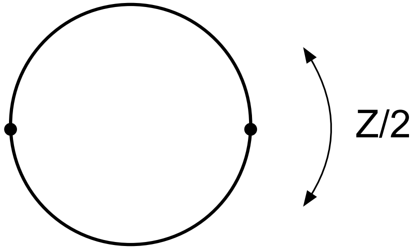
Composition

$$(\alpha, \gamma) \circ (\beta, \zeta) = (\alpha\beta, \gamma * \alpha\zeta)$$

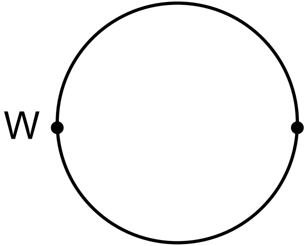
where $*$ is contatenation



Example: S^1 as $\mathbb{Z}/2$ space



Example: S^1 as $\mathbb{Z}/2$ space



W appears as two different objects:

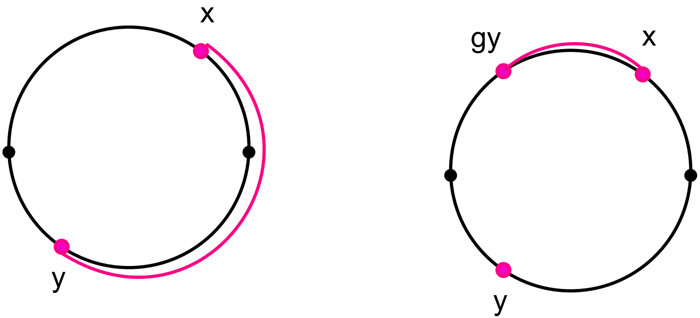
$W \in X^e$ gives $(G/e, W)$

$W \in X^G$ gives $(G/G, W)$

Every point comes tagged with specific fixed set; I will often abuse notation and suppress the G/H

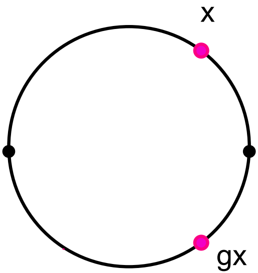
Example: S^1 as $\mathbb{Z}/2$ space

Two paths from x to y:

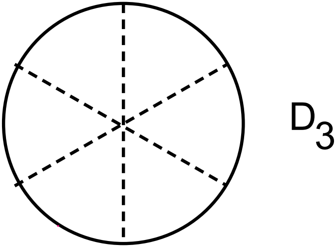


Example: S^1

There is a constant path from x to gx:

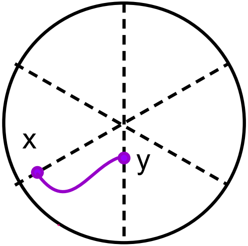


Example: D_3



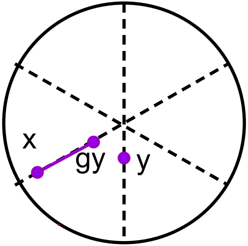
Example: D_3

A path from $(G/e, x)$ to $(G/e, y)$

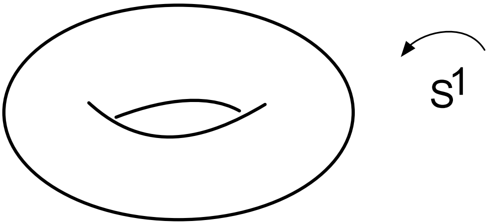


Example: D_3

A path from $(G/H, x)$ to $(G/K, y)$
Here the path must stay in X^H

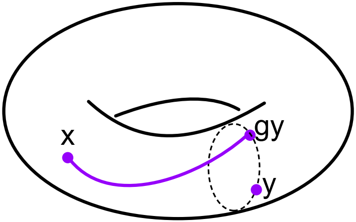


Example: T^2



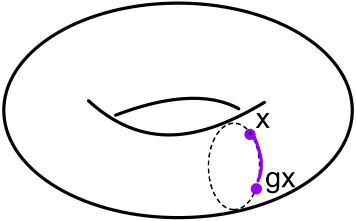
Example: T^2

A path from x to y:

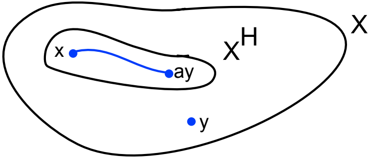


Example: T^2

A loop from x to x:



Jumping Between Fixed Sets



- If $x \in X^K$, then there is constant path $(G/H, x) \to (G/K, x)$ for $H \le K$
- If $\alpha y \in X^H$ then $y \in X^{\alpha H \alpha^{-1}}$
- There exists a path $x \longrightarrow y$ when

  $x \in X^H$
  $y \in X^K$
  $H \le \alpha K \alpha^{-1}$
  there is a path in X^H from x to αy

Making this Categorical: Grothendieck Construction

Let $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ be a contravariant functor.
The Grothendieck category

$$\int_{\mathcal{C}} \mathcal{F}$$

is defined by:

- An object is a pair (C, x) with $C \in \mathcal{C}_0$ and $x \in \mathcal{F}(C)_0$.
- An arrow $(g, \psi) \colon (C, x) \to (C', x')$ is a pair with $g \colon C \to C'$ in $\mathcal{C}_1$ and $\psi \colon x \to \mathcal{F}(g)(x')$ in $\mathcal{F}(C)_1$.
- Composition is defined by

$$\left((C, x) \xrightarrow{(g,\psi)} (C', x') \xrightarrow{(g',\psi')} (C'', x'') \right)$$

$$= (C, x) \xrightarrow{(g'g, \mathcal{F}(g)(\psi') \circ \psi)} (C'', x'')$$

Functor from What?
Orbit Category

$\mathcal{O}_G$ has

- objects: G/H for closed subgroups of G
- arrows: G-maps $G/H \to G/K$ defined by $H \to \alpha K$ for $\alpha \in G$ such that $H \le \alpha K \alpha^{-1}$.
- $\alpha, \beta \in G$ define the same map when $\alpha K = \beta K$
- Thus maps $G/H \to G/K$ are defined by elements $\alpha \in (G/K)^H$.

Orbit Category

$\mathcal{O}_G$ organizes fixed sets:

- A G-map $G/H \xrightarrow{x} X$ is defined by $H \to x$ for $x \in X^H$; then $gH \to gx$.
- X defines a contravariant functor $\Phi : \mathcal{O}_G \to \text{Spaces}$: $G/H \to X^H$
- if $\alpha : G/H \to G/K$, and $G/K \xrightarrow{x} X$ we can compose $G/H \xrightarrow{\alpha} G/K \xrightarrow{x} X$. This is defined by $H \to \alpha K \to \alpha x$, so $x \circ \alpha$ is just αx.

Interpret Fundamental Category as a Grothendieck Construction

Let $\underline{\Pi}_X(G/H) = \Pi(X^H)$ the fundamental groupoid of X^H: $\underline{\Pi}_X(G/H)$ has

- objects given by points $x \in X^H$
- arrows given by homotopy classes of paths in X^H

For $\alpha\colon G/H \to G/K$, define a functor $\Pi(X^K) \to \Pi(X^H)$:

- $x \in X^K$ goes to $\alpha x \in X^H$
- γ in X^K goes to $\alpha\gamma$ in X^H.

Then

$$\Pi_1(G, X) = \int_{\mathcal{O}_G} \underline{\Pi}_X.$$

- objects are pairs $(G/H, x)$ with $x \in X^H$
- arrows are pairs

$$(\alpha, \gamma)\colon (G/H, x) \to (G/K, y),$$

where $\alpha\colon G/H \to G/K$ and γ is a path from x to αy in X^H

Discrete $\Pi_1^d(G, X)$

- Objects of $\pi_1^d(G, X)$ are $x \in X^H$
- arrows are equivalence classes of maps $\gamma : x \to \alpha y$
- $(\alpha, \gamma) \simeq (\beta, \zeta)$ when there exists

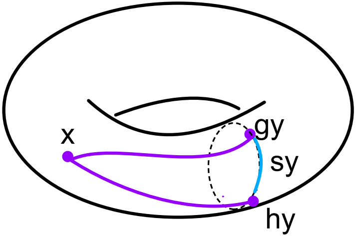
$$\sigma : G/H \times I \to G/K$$

from α to β and $\Lambda : I \times I \to X^H$ such that

$$\Lambda(0, t) = x$$
$$\Lambda(1, t) = \sigma(t)y$$
$$\Lambda(s, 0) = \gamma(s)$$
$$\Lambda(s, 1) = \zeta(s)$$

Example: T^2

We have a 2-cell $s : g \to h$ from (g, γ) to (h, γ')



$$\gamma * sy \simeq \gamma'$$

Making this Categorical: Grothendieck 2 Category

[Bakovic, Buckley]

- $\mathcal{C}$ a 2-category,
- $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ a contravariant 2-functor

$\int_{\mathcal{C}} \mathcal{F}$ is a 2-category defined by:

- An object is a pair (C, x) with $C \in \mathcal{C}_0$ and $x \in \mathcal{F}(C)_0$
- An arrow $(g, \psi) \colon (C, x) \to (C', x')$ is a pair with $g \colon C \to C'$ in $\mathcal{C}_1$ and $\psi \colon x \to \mathcal{F}(g)(x')$ in $\mathcal{F}(C)_1$
- A 2-cell $\alpha \colon (g, \psi) \Rightarrow (g', \psi') \colon (C, x) \rightrightarrows (C', x')$ is a 2-cell $\alpha \colon g \Rightarrow g'$ in $\mathcal{C}$ such that the diagram commutes in $\mathcal{F}(C)$:

$$\begin{array}{ccc} x & \xrightarrow{\psi} & \mathcal{F}(g)(x') \\ \Big\| & & \Big\downarrow \mathcal{F}(\alpha)_{x'} \\ x & \xrightarrow[\psi']{} & \mathcal{F}(g')(x') \end{array}$$

Creating $\Pi^d(G, X)$ as a Grothendieck 2-category

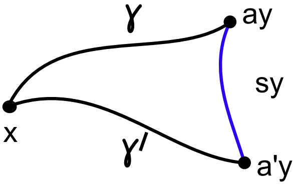
We saw earlier that we have a functor

- $\underline{\Pi}_X(G/H)$ is a category (a groupoid)
- $\alpha\colon G/H \to G/K$ defines a functor $\Pi(X^K) \to \Pi(X^H)$ (acting by α)
- If $\sigma : \alpha \longrightarrow \alpha'$ is a path in $\mathcal{O}_G(G/H, G/K)$ define a natural transformation with components given by arrows σx for $x \in X^K$.

  Create 2-category

$$\Pi_1(G, X) = \int_{\mathcal{O}_G} \underline{\Pi}_X.$$

  2-cells are exactly what tom Dieck mods out:



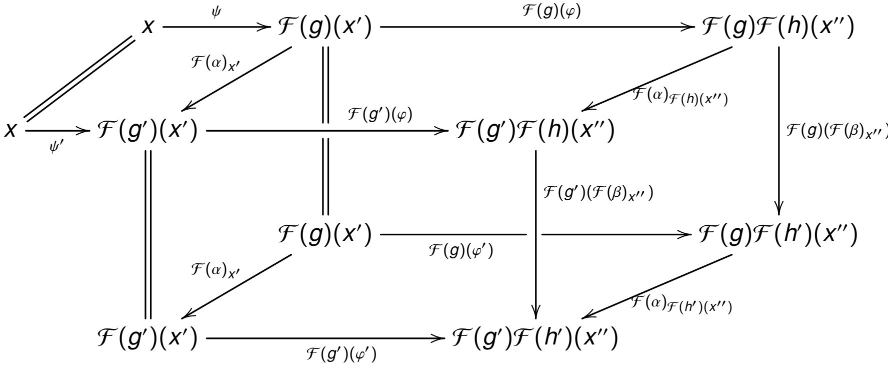
Composing 2-cells:

Category theory says:

Let $\alpha\colon (g,\psi) \Rightarrow (g',\psi')\colon (C,x) \rightrightarrows (C',x')$ and
$\beta\colon (h,\theta) \Rightarrow (h',\theta')\colon (C',x') \rightrightarrows (C'',x'')$ be 2-cells in $\int_{\mathcal{C}} \mathcal{F}$:

$$\begin{array}{ccc} x & \xrightarrow{\psi} & \mathcal{F}(g)(x') \\ \| & & \downarrow{\scriptstyle \mathcal{F}(\alpha)_{x'}} \\ x & \xrightarrow[\psi']{} & \mathcal{F}(g')(x') \end{array} \quad \text{in } \mathcal{F}(C), \text{ and} \qquad \begin{array}{ccc} x' & \xrightarrow{\varphi} & \mathcal{F}(h)(x'') \\ \| & & \downarrow{\scriptstyle \mathcal{F}(\beta)_{x''}} \\ x' & \xrightarrow[\varphi']{} & \mathcal{F}(h')(x'') \end{array} \quad \text{in } \mathcal{F}(C').$$

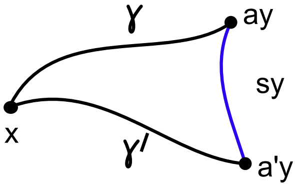
Composing 2-cells:

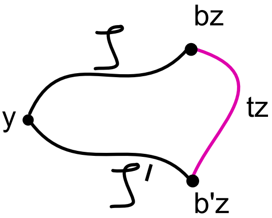


to get $\mathcal{F}(\beta \circ \alpha)$.

Composing 2-cells

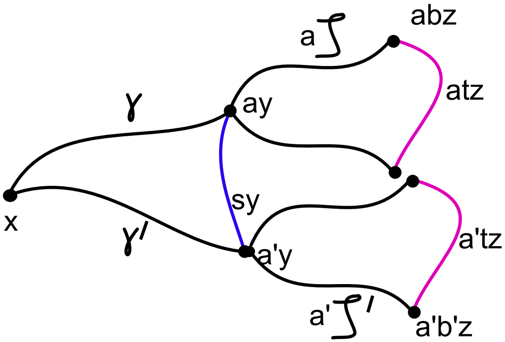
$\gamma : x \to \alpha y$ and $\gamma' : x \to \alpha' y$ with a 2-cell $s : \alpha \to \alpha'$
and
$\zeta : y \to \beta z$ and $\zeta' : x \to \beta' z$ with a 2-cell $t : \beta \to \beta'$:





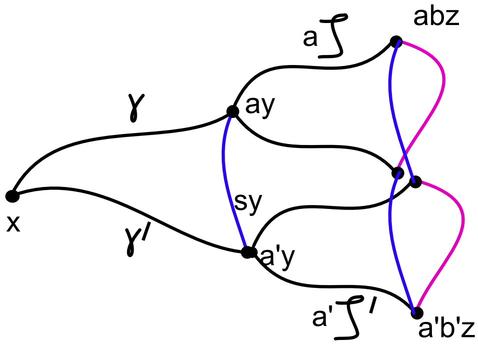
Composing 2-cells

We want to get a 2-cell from $\gamma * \alpha\zeta$ to $\gamma' * \alpha'\zeta'$:



Composing 2-cells

Fill in $s\beta z : \alpha\beta z \to \alpha'\beta z$ and $s\beta' z : \alpha\beta' z \to \alpha'\beta' z$

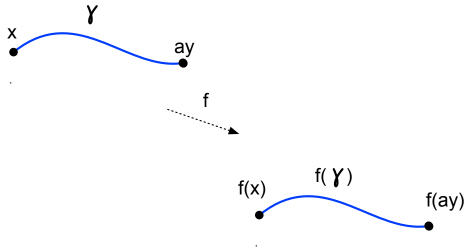


Then $\alpha tz * s\beta' z \simeq s\beta z * \alpha' tz$ and this gives the required 2-cell.

Functoriality

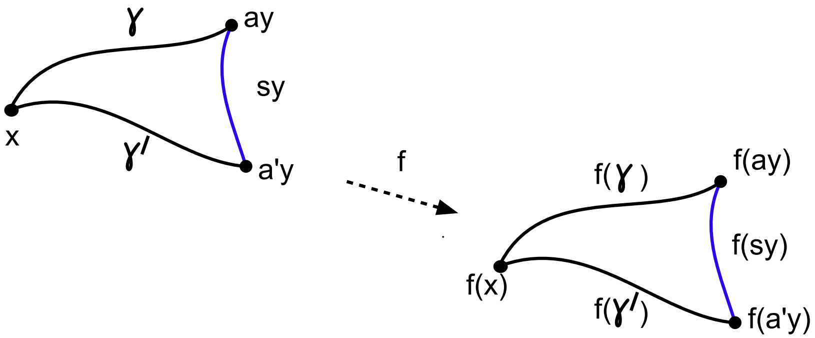
Suppose X_1 is a G_1-space, and X_2 is a G_2-space. A morphism is given by a group homomorphism $\varphi : G_1 \to G_2$ and an equivariant map $f : X_1 \to X_2$ such that $f(gx) = \varphi(g)f(x)$.
Then we get $\Pi(\varphi, f) : \Pi_{G_1}(X_1) \to \Pi_{G_2}(X_2)$

- Objects: $F(G_1/H, x) = (G_2/\varphi(H), f(x))$.
  If $x \in X_1^H$, then $f(x) \in X_2^{\varphi(H)}$.
- Arrows: If $\gamma : x \to \alpha y$ in X^H, define
  $F(\gamma) = f(\gamma) : f(x) \longrightarrow f(\alpha y) = \varphi(\alpha)f(y)$.



Functoriality

2-cells:



Natural Transformations

Natural transformations between equivariant maps:

$$r : X_1 \to G_2$$

denote $r(x) = r_x$, such that

$$r_x f_1(x) = f_2(x)$$

Naturality:

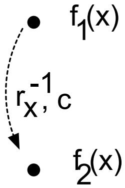
$$\begin{array}{ccc}
f_1(x) & \xrightarrow{\quad r_x \quad} & f_2(x) \\
\Big\downarrow {\scriptstyle \varphi(g)} & & \Big\downarrow {\scriptstyle \varphi_2(g)} \\
f_1(gx) = \varphi_1(g) f_1(x) & \xrightarrow[\quad r_{gx} \quad]{} & f_2(gx) = \varphi_2(g) f_2(x)
\end{array} \quad .$$

So

$$r_{gx} \varphi_1(g) = \varphi_2(x) r_x$$

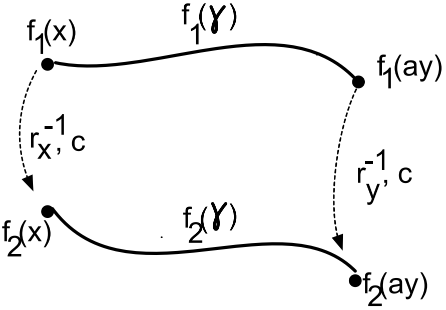
Natural Transformations

Make Π into a 2-functor: Suppose r is a natural transformation from (φ_1, f_1) to (φ_2, f_2).
Given $(G/H, x)$ with $x \in X^H$, assign the constant path $c_{f_1(x)}$ from $f_1(x) = r_x^{-1} f_2(x)$ to $f_2(x)$



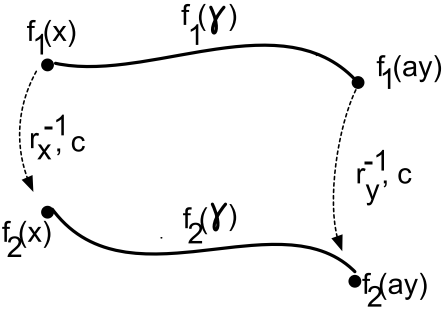
Natural Transformations

Is this natural?
let $\gamma : x \to y$ be an arrow of $\Pi_{G_1}(X_1)$
given by a path $\gamma : x \to \alpha y$.
Consider naturality square of arrows $f_1(x) \to f_2(y)$:



Natural Transformations

Compare compositions:



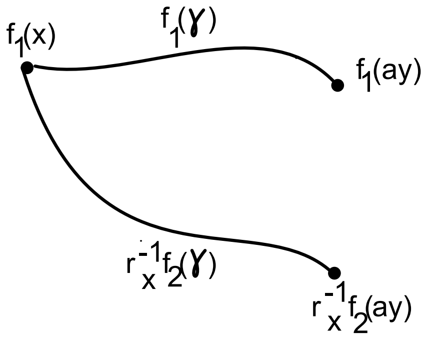
$$c_{f_1(x)} * r_x^{-1} f_2(\gamma) \simeq r_x^{-1} f_2(\gamma)$$

and

$$f_1(\gamma) * c_{f_1(\alpha y)} \simeq f_1(\gamma)$$

Natural Transformations

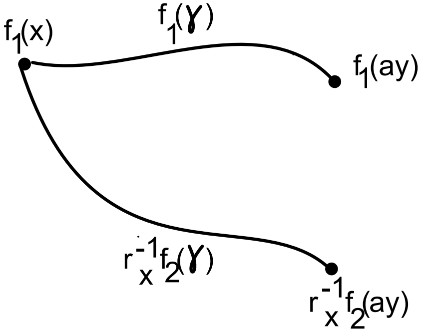
Compare compositions:



These are not the same.

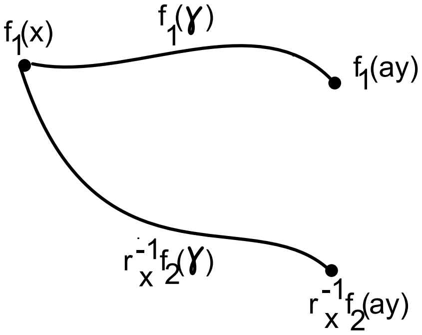
Pseudo Natural Transformation

For every morphism $\gamma : x \to \alpha y$, we assign a 2-cell to fill in the diagram (and satisfy required coherence.)



Psuedo Natural Transformations

Use equviariance and naturality to rewrite the ends of this:



$$f_1(\alpha y) = \varphi_1(\alpha) f_1(y) = \varphi_1(\alpha) r_y^{-1} f_2(y) = r_{\alpha y}^{-1} \varphi_2(\alpha) f_2(y)$$

and

$$r_x^{-1} f_2(\alpha y) = r_x^{-1} \varphi_2(\alpha) f_2(y)$$

Psuedo Natural Transformations

Use equviariance and naturality to rewrite the ends of this:

$$\begin{array}{ccc} f_1(x) & \xrightarrow{\;f_1(\gamma)\;} & r_{\alpha y}^{-1}\varphi_2(\alpha) f_2(y) \\ & \searrow^{\,r_x^{-1} f_2(\gamma)} & \\ & & r_x^{-1}\varphi_2(\alpha) f_2(y) \end{array}$$

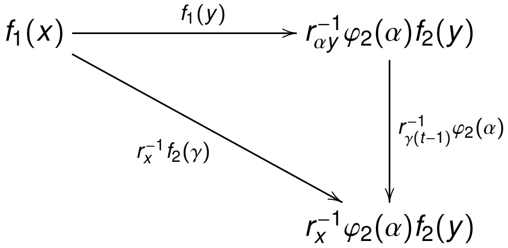
Remember that $\gamma : x \longrightarrow \alpha y$.

Pseudo Natural Transformation

Define

$$s(\gamma)(t) = s(t) = r^{-1}_{\gamma(t-1)}\varphi_2(\alpha).$$

so that



Where is this going?

Many orbifolds are represented by compact Lie group actions

This representation is not unique - Morita equivalence

All Morita equivalences are given by equivariant maps of two very specific types

Goal: show that the discrete fundamental group category is an orbifold invariant

- Igor Bakovic, *Fibrations of bicategories*, http://www.irb.hr/korisnici/ibakovic/groth2fib.pdf
- Mitchell Buckley, Fibred 2-categories and bicategories, *Journal of Pure and Applied Algebra* **218** (2014), pp. 1034–1074.
- T. tom Dieck, *Transformation Groups*, Walter de Gruyter (1987).