# Kan-injectivity and KZ-monads

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IPV / CMUC

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[M. Carvalho, L.S., 2011] :

Kan-injectivity/KZ-monads enjoys many features resembling Orthogonality/Idempotent monads

- M. Carvalho, L. S., Order-preserving reflectors and injectivity, TA, 2011
- J. Adámek, L. S., J. Velebil, Kan injectivity in order-enr. cats., MSCS, 2015
- M. Carvalho, L. S., On Kan-injectivity of locales and spaces, ACS, 2017
- L. S., A calculus of lax fractions, JPAA, 2017
- J. Adámek, L. S., KZ-monadic categories and their logic, TAC, 2017
- D. Hofmann, L. S., Aspects of algebraic algebras, LMCS, 2017

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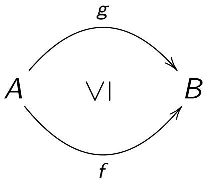
- M. M. Clementino, F. Lucatelli, J. Picado: joint work in progress

1. Kan-injectivity and KZ-monads
2. In locales and topological spaces
3. Lax fractions
4. Kan-injective subcategory problem

Most of the time, the setting is

order-enriched categories



Monad $\mathbb{T} = (T, \eta, \mu)$ of Kock-Zöberlein type: $T\eta \leq \eta T$

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KZ-monadic subcategory of $\mathcal{X}$= Eilenberg-Moore category of a KZ-monad over $\mathcal{X}$

Full reflective subcategory of $\mathcal{X}$ = Eilenberg-Moore category of an idempotent monad over $\mathcal{X}$ ($T\eta = \eta T$)

$\mathbb{T} = (T, \eta, \mu)$ idempotent:
$A \in \mathcal{X}^{\mathbb{T}}$ iff it is orthogonal to all $\eta_X$, i.e.,

$$\mathcal{X}(TX, A) \xrightarrow{\mathcal{X}(\eta_X, A)} \mathcal{X}(X, A) \quad \text{is an isomorphism.}$$

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$g$ is a right adjoint retraction if
there is an adjunction $(id, \beta) : f \dashv g$

In order enriched categories:

$$gf = id \text{ and } fg \leq id$$

$A$ is (left) Kan-injective wrt $h : X \to Y$ if

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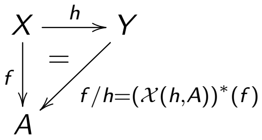
Equivalently: for all $f : X \to A$, there exists a left Kan extension of $f$ along $h$ of the form $\mathrm{Lan}_h(f) = (f/h, id)$.



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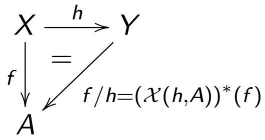


$k : A \to B$ is (left) Kan-injective wrt $h : X \to Y$, if $A$ and $B$ are so, and $k$ preserves the left Kan extension of every $f : X \to A$ along $k$.

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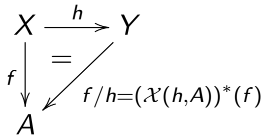


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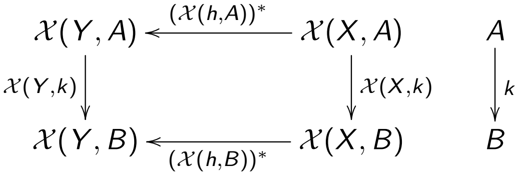

For $\mathcal{H} \subseteq \mathsf{Mor}(\mathcal{X})$,

$\underbrace{\mathsf{KInj}(\mathcal{H})}$ := (locally full) subcategory of objects and morphisms Kan-injective wrt all $h \in \mathcal{H}$

(Left) Kan-injective subcategory

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For $\mathbb{T} = (T, \eta, \mu)$ a KZ-monad over $\mathcal{X}$ order-enriched,

$\mathcal{X}^{\mathbb{T}} = \mathsf{KInj}(\{\eta_X \mid X \in \mathcal{X}\})$.

$\mathcal{A}$ a (locally full) subcategory of $\mathcal{X}$

$\mathcal{A}$ is closed under left adjoint retractions, if, for every commutative diagram

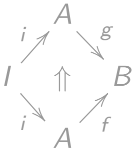
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ X & \xrightarrow[g]{} & Y \end{array}$$

with $q$ and $q'$ left adjoint retractions, whenever $f \in \mathcal{A}$, then $g \in \mathcal{A}$.

$\mathcal{A}$ a (locally full) subcategory of $\mathcal{X}$

$\mathcal{A}$ is an inserter-ideal, provided that, for every inserter diagram

$$I \xrightarrow[i=\mathrm{ins}(f,g)]{} A \overset{g}{\underset{f}{\rightrightarrows}} B$$




$f \in \mathcal{A} \implies i \in \mathcal{A}$.

**Theorem ([CS, 2011], [ASV, 2015])**

*Given $\mathcal{H} \subseteq Mor(\mathcal{X})$, $\mathsf{KInj}(\mathcal{H})$ is:*

1. *Closed under weighted limits, i.e., the inclusion functor $\mathsf{KInj}(\mathcal{H}) \hookrightarrow \mathcal{X}$ creates weighted limits;*
2. *An inserter-ideal;*
3. *Closed under left adjoint retractions.*

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## Corollary

*Every KZ-monadic subcategory enjoys properties 1, 2 and 3 above.*

**Theorem ([ASV, 2015])**

*Let $\mathcal{X}$ have inserters. A reflection of $\mathcal{X}$ in a subcategory $\mathcal{A}$ is of Kock-Zöberlein type (i.e. it induces a KZ-monad), iff $\mathcal{A}$ is an inserter-ideal of $\mathcal{X}$.*

**Theorem ([CS, 2011])**

*Let $\mathcal{A}$ be a (locally full) subcategory of $\mathcal{X}$. The inclusion functor $E : \mathcal{A} \hookrightarrow \mathcal{X}$ is a right adjoint which induces a KZ-monad over $\mathcal{X}$, iff for every $X \in \mathcal{X}$, there is an arrow $\eta_X : X \to \overline{X}$ with $\overline{X} \in \mathcal{A}$ such that:*

(i) $\mathcal{A} \subseteq \mathsf{KInj}(\{\eta_X \,|\, X \in \mathcal{X}\})$ *and, for every* $f : X \to A$ *with* $A$ *in* $\mathcal{A}$ $f/\eta_X \in \mathcal{A}$.

(ii) $\eta_X$ *is dense, i.e.,* $\eta_X/\eta_X = id_{\overline{X}}$, $X \in \mathcal{X}$.

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In the setting of 2-categories:

[F. Marmolejo, R. Wood, Kan extensions and lax idempotent pseudomonads, TAC, 2012]

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(ii) $\eta_X$ *is dense, i.e.,* $\eta_X/\eta_X = id_{\overline{X}}$, $X \in \mathcal{X}$.

*Furthermore, under the above conditions, $\mathcal{A}$ is a KZ-monadic subcategory of $\mathcal{X}$ iff it is closed under left adjoint retractions.*

| | | |
|---|---|---|
| Eilenberg-Moore category | = | closure under left adjoint retractions of the Kleisli category |

$\mathbb{T}$ = filter monad on $\mathsf{Top}_0$

$$\mathsf{Top_0}_{\mathbb{T}} \hookrightarrow \mathrm{Idemp}(\mathsf{Top_0}_{\mathbb{T}}) \hookrightarrow \mathrm{Lim}(\mathsf{Top_0}_{\mathbb{T}}) \hookrightarrow {\mathsf{Top}_0}^{\mathbb{T}} \longrightarrow \mathsf{Top}_0$$

| $\mathrm{Idemp}(\mathsf{Top_0}_{\mathbb{T}})$ | $\mathrm{Lim}(\mathsf{Top_0}_{\mathbb{T}})$ | ${\mathsf{Top}_0}^{\mathbb{T}}$ |
|---|---|---|
| $\Vert$ | $\Vert$ | $\Vert$ |
| Idemp. split compl. of Kleisli | Closure under w. lims. of Kleisli | Closure of Kleisli under left adjoint retractions |
| $\Vert$ | $\Vert$ | $\Vert$ |
| Algebraic lattices whose compacts form a coframe | Algebraic lattices | Continuous lattices |

[HS, 2017]

$\mathcal{A}$ subcategory of $\mathcal{X}$

$\mathcal{A}^{\mathsf{KInj}} := \{h \in \mathsf{Mor}(\mathcal{X}) \,|\, \mathcal{A} \text{ Kan-injective wrt } h\}$

Galois connection:

$$\mathcal{H} \longmapsto \mathsf{KInj}(\mathcal{H})$$
$$\mathcal{A}^{\mathsf{KInj}} \longleftarrow\!\!\dashv \mathcal{A}$$

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In case $\mathcal{A}$ is an Eilenberg-Moore category of a KZ-monad $T$

$$\mathcal{A}^{\mathsf{KInj}} = \{f \mid f \text{ is a } \underbrace{T\text{-embedding}}_{Tf \text{ is a left adjoint section}}\}$$

[D. Scott, LN, 1972]:

In $\mathrm{Top}_0$, continuous lattices = spaces injective wrt embeddings

[P. Johnstone, JPAA, 1981]:

In Loc,
stably locally compact locales = retracts of coherent locales
= locales injective wrt flat embeddings

M. Escardó, in 1990's:

Several examples of

injective objs. = EM-algebras of a KZ-monad

$\mathsf{Loc} = \mathsf{Frm}^{\mathsf{op}}$

Locale = frame = complete lattice $L$ with $(\bigvee A) \wedge b = \bigvee_{a \in A} (a \wedge b)$

Localic map = infima-preserving map $f : L \to M$ with $f^* : M \to L$ preserving finite meets

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$f : L \to M$ is $n$-flat, if $f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)$, for $|I| \leq n$.

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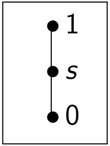
(0-flat =) 1-flat = dense ($f(0) = 0$)

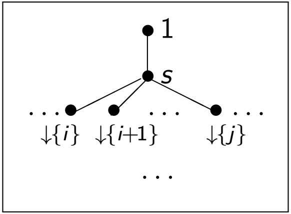
2-flat = flat

For every cardinal $n$,

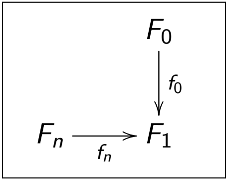
$$F_n = \text{free frame generated by the set } n$$

$$F_n = \big(\{\text{downsets of } (\{\text{finite subsets of } X\}, \supseteq)\,,\subseteq\big)$$

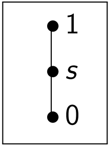
$F_1 =$ 

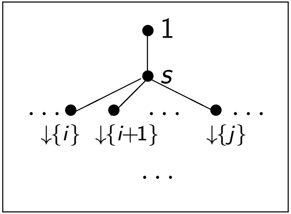

$F_n =$ 


$\downarrow\{i\} = \{A \subseteq n \mid A \text{ fin.}, i \in A\}$

$\mathcal{D}_n =$ 


with $f_n(1) = 1, \; f_n(s) = s$, and $f_n(x) = 0$ otherwise

$F_1 =$ 


$F_n =$ 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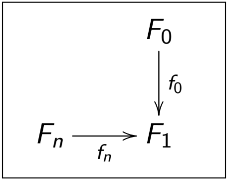
$\mathcal{D}_n =$ 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**Theorem ([CS, 2017])**

- *Embeddings* $= F_1^{\mathsf{KInj}}$
- *n-flat embeddings* $= \mathcal{D}_n^{\mathsf{KInj}}$

$\mathcal{D} = \bigcup_{n \in \mathsf{Card}} \mathcal{D}_n$ is a subcategory of Loc made of spatial locales.

## Corollary

Loc *is the Kan-injective hull of a subcategory made of spatial locales:*

$$\mathsf{Loc} = \mathsf{KInj}\left(\mathcal{D}^{\mathsf{KInj}}\right)$$

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*Proof.*

$$\begin{aligned}
\mathcal{D}^{\mathsf{KInj}} &= \bigcap_{n \in \mathsf{Card}} \mathcal{D}_n^{\mathsf{KInj}} \\
&= \{\{f \in \mathsf{Loc} \mid f_* \in \mathsf{Loc} \text{ and } f_* f = \mathrm{id}\} \\
&= \underbrace{\{f \in \mathsf{Loc} \mid f \text{ is a left adjoint section in Loc}\}}_{\mathcal{H}}
\end{aligned}$$

Thus, $\mathsf{KInj}(\mathcal{H}) = \mathsf{Loc}$.

$L \in \mathsf{Loc}$

Given $n$,

$$G_n L := \{U \subseteq L \mid U = \downarrow U, U \text{ closed under } \bigvee_I, |I| \leq n\} \quad \text{with } \subseteq$$

$$G_n : \mathsf{Loc} \to \mathsf{Loc}$$

gives rise to the functor part of a KZ-monad.

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$a \ll_n b$, if, $\forall U \in G_n L$,

$$b \leq \bigvee U \;\Rightarrow\; a \in U$$

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$a \ll_n b$, if, $\forall U \in G_n L$,

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$L$ is stably locally $n$-compact if

- $\forall a \in L,\; a = \bigvee_{x \ll_n a} x$
- $\forall x,\, a,\, b,\; (x \ll_n a,\; x \ll_n b) \Rightarrow x \ll_n a \wedge b$
- $1 \ll_n 1$

$\mathsf{SLComp_n} =$ category of stably locally $n$-compact locales and localic maps $f$ such that $f^*$ preserves $\ll_n$

**Theorem ([CS, 2017])**

*For every $n$, $\mathsf{SLComp_n}$ is a KZ-monadic subcategory, and it is the Kan-injective hull of $\mathcal{D}_n$, i.e.,*

$$\mathsf{SLComp_n} = \mathsf{KInj}\left(\mathcal{D}_n^{\mathsf{KInj}}\right).$$

$$\begin{array}{ccc} \mathsf{Top}_0 & \xrightarrow{\mathsf{Lc}} & \mathsf{Loc} \\ X & \longmapsto & \Omega(X) \\ f & \longmapsto & (f^{-1})_* \end{array}$$

$\mathsf{Lc} \dashv \mathsf{Sp} : \mathsf{Loc} \to \mathsf{Top}_0$ order-enriched adjunction

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$\mathsf{Lc} \dashv \mathsf{Sp} : \mathsf{Loc} \to \mathsf{Top}_0$ order-enriched adjunction

$X \xrightarrow{f} Y$ is an embedding in $\mathsf{Top}_0$ iff $\mathsf{Lc}(f)$ is an embedding in Loc

is dense in $\mathsf{Top}_0$ iff $\mathsf{Lc}(f)$ is dense in Loc

is $n$-flat in $\mathsf{Top}_0$ iff $\mathsf{Lc}(f)$ is $n$-flat in Loc

**Lemma**

*Let $F \dashv G : \mathcal{A} \to \mathcal{X}$ be an order-enriched adjunction.*
*Then, given $h$ in $\mathcal{X}$ and an object $A$ (resp., a morphism $f$) in $\mathcal{A}$,*

$$A \text{ (resp., } f\text{) is Kan-injective wrt } Fh$$
$$\Updownarrow$$
$$GA \text{ (resp., } Gf\text{) is Kan-injective wrt } h.$$

*Proof.* Immediate from the natural isomorphism

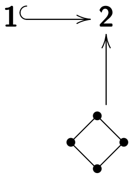
$$\mathcal{A}(FX, A) \cong \mathcal{X}(X, GA).$$

**Corollary ([CS, 2017])**

*In* $\mathsf{Top}_0$*:*

- *Embeddings are precisely the morphisms wrt which the Sierpiński space is Kan-injective.*
- *n-flat embeddings are precisely the morphisms wrt which* $Sp[\mathcal{D}_n]$ *is Kan-injective.*

In $\mathsf{Top}_0$:

| $\mathcal{A}$ | $\mathcal{A}^{\mathsf{KInj}}$ | $\mathsf{KInj}\,(\mathcal{A}^{\mathsf{KInj}})$ (KZ-monadic) |
|---|---|---|
| $\mathbf{2} =$ Sierpiński | embeddings | continuous lattices & maps pres. all $\bigwedge$ and $\bigvee^{\uparrow}$ |
| $\mathbf{1} \hookrightarrow \mathbf{2}$ | dense embeddings | Scott conts. lats. & maps pres. $\bigwedge$ $(\neq \emptyset)$ and $\bigvee^{\uparrow}$ |
|  | flat embeddings | stably locally compact spaces & convenient maps |

We need Kan-injectivity w.r.t. squares

full reflective subcategory: the Kleisli category of the idemp. monad $T$ is a category of fractions of $\underbrace{\{h \mid Th \text{ is an iso}\}}_{= \mathcal{A}^{\mathsf{Orth}}}$

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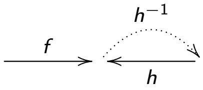
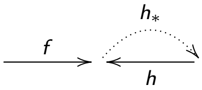
fraction: $h^{-1} \cdot f$




lax fraction: $h_* \cdot f$



$\mathcal{A}^{\mathsf{Orth}}$ closed under colimits in $\mathcal{X}^{\to}$

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Applications:

- $\mathcal{A}^{\mathsf{Orth}}$ admits a calculus of fractions
- an affirmative answer to the Orthog. Subcat. Problem [Gabriel, Ulmer, 1971] [Kelly, 1980]

$\vdots$

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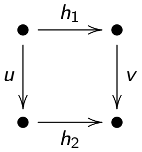
$\vdots$

What about $\mathcal{A}^{\mathsf{KInj}}$ ?

$$S = \quad \begin{array}{ccc} \bullet & \xrightarrow{h_1} & \bullet \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle v} \\ \bullet & \xrightarrow[h_2]{} & \bullet \end{array}$$

is a square in $\mathcal{X}$. It represents the morphism

$$(u, v) : h_1 \to h_2 \quad \text{in } \mathcal{X}^{\to}.$$

$S =$ 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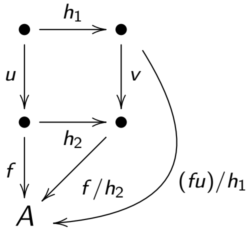
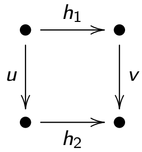
$A$ is Kan-injective wrt S if it is Kan-injective wrt $h_1$ and $h_2$ and, for every $f$, $(fu)/h_1 = (f/h_2)v$:



$S =$ 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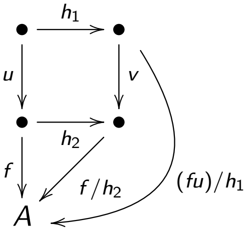



$k : A \to B$ is Kan-injective wrt S if it is Kan-injective wrt $h_1$ and $h_2$.

$\mathcal{A}^{\underline{\mathsf{KInj}}} =$ subcategory of $\mathcal{X}^{\to}$ of morphisms and squares wrt which $\mathcal{A}$ is Kan-injective

$\mathcal{A}^{\underline{\mathsf{KInj}}}$ = subcategory of $\mathcal{X}^{\rightarrow}$ of morphisms and squares wrt which $\mathcal{A}$ is Kan-injective

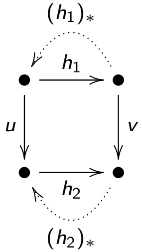
**Theorem**

*Let $\mathcal{X}$ have weighted colimits.*
*$\mathcal{A}^{\underline{\mathsf{KInj}}}$ is closed under weighted colimits in $\mathcal{X}^{\rightarrow}$. And it is a coinserter-ideal.*

A morphism $h$ as a square:

$$S(h) = \begin{array}{ccc} \bullet & \xrightarrow{h} & \bullet \\ \Big\| & & \Big\| \\ \bullet & \xrightarrow[h]{} & \bullet \end{array}$$

A split square is a square




with $h_1$ and $h_2$ left adjoint sections and $(h_2)_* v = u(h_1)_*$.

A square $S$ is a split square iff $\mathrm{KInj}(S) = \mathcal{X}$.

Let $\mathcal{H}$ be a class of squares of $\mathcal{X}$.

A category of lax fractions for $\mathcal{H}$ is a functor $F : \mathcal{X} \to \mathcal{X}[\mathcal{H}_*]$ such that:

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   is a split square.
2. If $G : \mathcal{X} \to \mathcal{C}$ is another functor under the above condition, then there is a unique functor $H : X[\mathcal{H}_*] \to \mathcal{C}$ such that $HF = G$.

**Theorem ([S, 2017])**

*Let $\mathcal{A}$ be a KZ-monadic subcategory of $\mathcal{X}$. Then the corresponding Kleisli category is a category of lax fractions for $\mathcal{H} = \mathcal{A}^{\underline{\mathsf{KInj}}}$.*

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Also in [S., 2017]: a calculus of lax fractions, via
a calculus of squares

In locally bounded categories, $\mathrm{Orth}(\mathcal{H})$ is reflective (for each set $\mathcal{H}$).

Each reflection of $X$ in $\mathrm{Orth}(\mathcal{H})$ is given by a convenient chain

$$X = X_0 \longrightarrow X_1 \longrightarrow \dots \longrightarrow X_i \longrightarrow \dots \longrightarrow X_\lambda$$

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Logic for Orthogonality

[J. Adámek, M. Hébert, L.S., The orthog. subcat. probl. ..., 2009]

[J. Adámek, M. Sobral, L.S., A logic of implications ..., 2009]

Analogously, two related problems:

- Kan-Injective Subcategory Problem
- A Logic for Kan-injectivity

**Theorem ([ASV, 2015])**

*In a locally bounded order-enriched category,* KInj($\mathcal{H}$) *is KZ-monadic, for every set* $\mathcal{H}$ *of morphisms.*

**Theorem ([ASV, 2015], [AS, 2017])**

*In a locally bounded order-enriched category,* $\mathsf{KInj}(\mathcal{H})$ *is KZ-monadic, for every set* $\mathcal{H}$ *of squares.*

To obtain a complete logic for Kan-injectivity, we need squares.

$\mathcal{X}$ is locally bounded, that is:

- it has weighted colimits;
- it has a proper f. s. $(E, M)$, i.e., $E \subseteq \mathrm{Epi}$, $M \subseteq \mathrm{OrderMono}$;
  $(mf \leq mg \;\Rightarrow\; f \leq g)$
- it is $E$-cowellpowered;
- every object $X$ has bound, i.e.,
  $\mathcal{X}(X, -)$ preserves $\lambda$-direced $M$-unions, for some $\lambda$.

## The reflection chain

Given a set $\mathcal{H}$ of squares, for every $X$, the chain

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \ldots \dashrightarrow X_i \dashrightarrow \ldots \quad (i \in \mathrm{Ord})$$

is constructed as follows:

Limit step $i$. $X_i = \operatorname*{colim}_{j<i} X_j$

.

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Isolated step $i \mapsto i+1$ ($i$ even).

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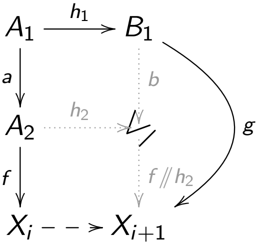
$$S = \begin{array}{ccc} A_1 & \xrightarrow{h_1} & B_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_2 & \xrightarrow[h_2]{} & B_2 \end{array}$$

$$\begin{array}{ccc} A_r & \xrightarrow{h_r} & B_r \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f/\!\!/h_r} \\ X_i & \dashrightarrow & X_{i+1} \end{array}$$

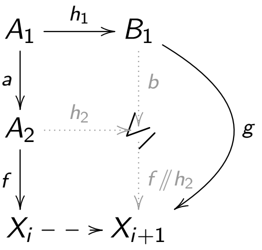
Wide pushout of all pushouts of $f$'s along $h_r$'s $(r = 1, 2)$ of $S \in \mathcal{H}$

Isolated step $i+1 \mapsto i+2$ .




$$\mathrm{coins}(f /\!\!/ h_2 \cdot b,\ g)$$

## Isolated step $i+1 \mapsto i+2$ .

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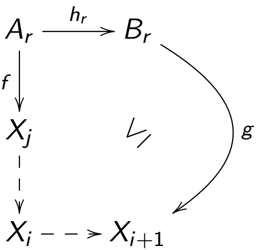


$$\mathrm{coins}(x_{j+1,i+1} \cdot f /\!\!/ h_r,\ g)$$

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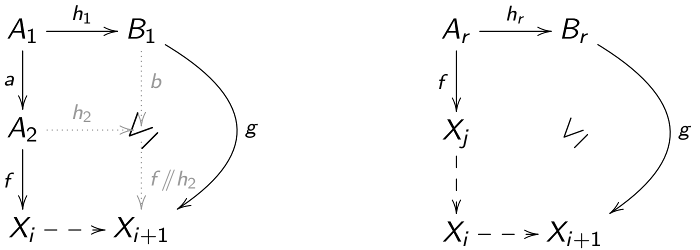



$\mathrm{coins}(f /\!/ h_2 \cdot b,\ g)$ $\qquad$ $\mathrm{coins}(x_{j+1,i+1} \cdot f /\!/ h_r,\ g)$

$X_{i+1} \dashrightarrow X_{i+2}$ is the wide pushout of all these coinserters for $S \in \mathcal{H}$, and possible $f$'s and $g$'s.

There is a cardinal $\lambda$, greater than the bounds of the objects appearing in the squares of $\mathcal{H}$, such that

$$X_0 \dashrightarrow X_\lambda$$

is a KZ-reflection in $\mathsf{KInj}(\mathcal{H})$.

Aim:

System of deduction rules such that,
for every set of squares $\mathcal{H}$ and every square $S$,

$$\mathcal{H} \vdash S \text{ iff } \mathcal{H} \models S$$

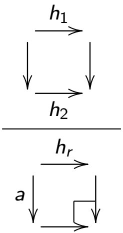
where $\mathcal{H} \models S$ means that $\mathsf{KInj}(\mathcal{H}) \subseteq \mathsf{KInj}(\{S\})$.

## Kan-Injectivity Deduction System

AXIOM $\dfrac{}{S}$ for split squares $S$

COMPOSITION $\dfrac{S_1 \quad S_2}{S}$ for a composite $S$, horizontal or vertical, of $S_1$ and $S_2$

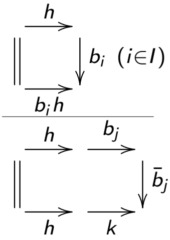
PUSHOUT



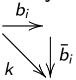

for a pushout of $h_r$, $r = 1$ or $2$, along an arbitrary morphism $a$

## Kan-Injectivity Deduction System

WIDE PUSHOUT



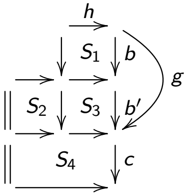

for any wide pushout




and $j \in I$

COINSERTER

$$\frac{S_1\,,\ S_2,\ \ S_3 \cdot S_2}{S_4}$$



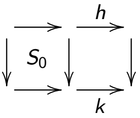

for $(b'b)h \leq gh$
and $c = \mathrm{coins}(b'b, g)$

## Kan-Injectivity Deduction System

RIGHT CANCELLATION

$$\frac{S\,,\ S(h)\,,\ S(k)}{S_0}$$

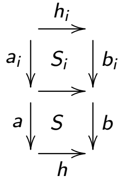
for $S =$




UPPER CANCELLATION

$$\frac{S_i\,,\ \overline{S}_i\ \ (i \in I)}{S}$$

for $\overline{S}_i =$




with $(b_i)_{i \in I}$ collectively epic

**Theorem**

*In any order-enriched locally bounded category, the Kan-injectivity Deduction System is sound and complete:*

$$\mathcal{H} \models S \quad \textit{iff} \quad \mathcal{H} \vdash S$$

**Theorem**

*In any locally bounded order-enriched category, for every set of squares $\mathcal{H}$, the class*

$$\{S \in Square(\mathcal{X}) \mid \mathcal{H} \models S\}$$

*is the smallest subcategory of $\mathcal{X}^{\rightarrow}$ containing $\mathcal{H}$ and all split squares, and closed under horizontal composition, weighted colimits, the coinserter rule, and right and upper cancellations.*

### Open question

Let $\mathcal{X}$ have weighted colimits.
Do Eilenberg-Moore categories of a KZ-monad over $\mathcal{X}$ have weighted colimits (at least under mild conditions)?