Day convolution, ∞-operads and Goodwillie calculus

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11 July 2018
Category Theory 2018
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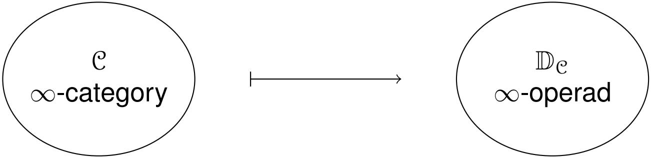
Overview



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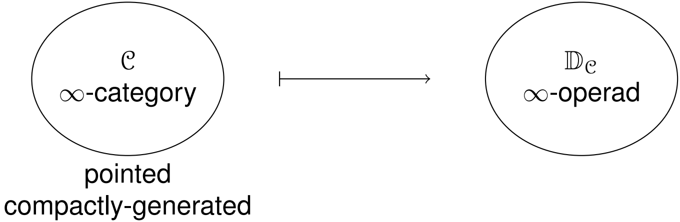
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∞-category
pointed
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$\mathbb{D}_{\mathcal{C}}$
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Examples:

Top_* (based spaces)

Lie operad

Overview

$$\mathcal{C} \longmapsto \mathbb{D}_{\mathcal{C}}$$

$\mathcal{C}$: ∞-category — pointed, compactly-generated

$\mathbb{D}_{\mathcal{C}}$: ∞-operad — stable, non-unital

Examples:

∞-category	∞-operad
Top_* (based spaces)	Lie operad
Sp (spectra)	trivial operad

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$\mathrm{Fun}(\mathbb{C}^{op}, \mathrm{Top}_*)$	?

Why I Care: Goodwillie Calculus

- Goodwillie associates to a functor $F : \mathcal{C} \to \mathcal{D}$, a Taylor tower:

$$F \to \cdots \to P_n F \to P_{n-1} F \to \cdots \to P_1 F$$

where $P_n F$ is the "best n-excisive approximation to F".

- The fibre $D_n F := \mathrm{fib}(P_n F \to P_{n-1} F)$ is n-homogeneous and can be classified in terms of a (symmetric multilinear) functor

$$\partial_n F : \mathrm{Sp}(\mathcal{C})^n \times \mathrm{Sp}(\mathcal{D})^{op} \to \mathrm{Sp}.$$

($\mathrm{Sp}(\mathcal{C})$ is the stabilization of $\mathcal{C}$) where, for $\mathcal{D} = \mathrm{Top}_*$:

$$D_n F(X) \simeq \Omega^\infty \partial_n F(X, \ldots, X; S^0)_{h\Sigma_n}.$$

- Ideally: (i) calculate $\partial_n F$, (ii) calculate $P_n F$, (iii) the tower converges to F.

Goodwillie Calculus and Operads

We need additional structure on the collection $(\partial_n F)_{n\geq 1}$ to tell us how to piece together the layers to calculate the tower $(P_n F)_{n\geq 1}$.

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Theorem (C., 2018)

1. *For a pointed compactly-generated ∞-category $\mathcal{C}$, there is a stable non-unital ∞-operad $\mathbb{D}_{\mathcal{C}}$ with:*
   - *colours: objects of* $\mathrm{Sp}(\mathcal{C})$*;*
   - *multi-morphism spectra*

$$\mathbb{D}_{\mathcal{C}}(X_1, \dots, X_n; Y) \simeq \partial_n I_{\mathcal{C}}(X_1, \dots, X_n; Y)$$

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2. *For $F : \mathcal{C} \to \mathcal{D}$ reduced and preserving filtered colimits, there is a bimodule $\mathbb{D}_F$ over the operads $\mathbb{D}_{\mathcal{C}}$ and $\mathbb{D}_{\mathcal{D}}$ with multi-morphism spectra*

   $$\mathbb{D}_F(X_1, \dots, X_n; Y) \simeq \partial_n F(X_1, \dots, X_n; Y)$$

   for $X_1, \dots, X_n \in \mathrm{Sp}(\mathcal{C})$, $Y \in \mathrm{Sp}(\mathcal{D})$.

Remarks

- This theorem was previously proved, jointly with Greg Arone, in the case where $\mathcal{C}$ and $\mathcal{D}$ are either Top_* or Sp. The construction here is very different though.
- When $\mathcal{C} = \mathrm{Top}_*$, the operad $\mathbb{D}_{\mathcal{C}}$ is analogous to the Lie operad in vector spaces. Algebras over this operad are therefore a kind of "spectral Lie algebras".
- The bimodule $\mathbb{D}_F$ contains "some" of the information needed to reconstruct the Taylor tower of F, but, in general, not all. Work in progress: there is a refinement of the constructions here in terms of pro-operads and pro-bimodules that contains all the information.

Day Convolution

- $\mathcal{C}$: pointed compactly-generated ∞-category
- $\mathrm{Fun}_*^{\omega}(\mathcal{C}, \mathrm{Sp})$: reduced filtered-colimit-preserving functors $\mathcal{C} \to \mathrm{Sp}$ with objectwise smash product $\bar{\wedge}$:

$$(F \mathbin{\bar{\wedge}} G)(X) := F(X) \wedge G(X)$$

  a (non-unital) symmetric monoidal ∞-category.
- Sp: spectra with smash product $\wedge$: a (non-unital) symmetric monoidal ∞-category.

Definition (Glasman, 2016)

Let $\otimes$ be the (non-unital) symmetric monoidal structure on

$$\mathrm{Fun}(\mathrm{Fun}_*^{\omega}(\mathcal{C}, \mathrm{Sp}), \mathrm{Sp})$$

given by Day convolution of $\bar{\wedge}$ and $\wedge$.

Examples of Day Convolution

1. Each $x \in \mathcal{C}$ determines an evaluation functor

$$\mathrm{ev}_x : \mathrm{Fun}_*^{\omega}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Sp}; \quad F \mapsto F(x).$$

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$$\mathrm{cr}_1(-)(x_1) \otimes \cdots \otimes \mathrm{cr}_1(-)(x_n) \simeq \mathrm{cr}_n(-)(x_1, \ldots, x_n).$$

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3. For $X_1, \ldots, X_n \in \mathrm{Sp}(\mathcal{C})$, we have a derivative functor
$$\partial_n(-)(X_1, \ldots, X_n; S^0) : \mathrm{Fun}_*^{\omega}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Sp}.$$
and
$$\partial_1(-)(X_1; S^0) \otimes \cdots \otimes \partial_1(-)(X_n; S^0) \simeq \partial_n(-)(X_1, \ldots, X_n; S^0).$$

The Construction of the ∞-operad $\mathbb{D}_{\mathcal{C}}$

- There is a corresponding symmetric monoidal structure $\otimes$ on the opposite category (Barwick-Glasman-Nardin, 2014)

$$\mathrm{Fun}(\mathrm{Fun}_*^{\omega}(\mathcal{C}, \mathrm{Sp}), \mathrm{Sp})^{op}.$$

- This symmetric monoidal structure can be viewed as a symmetric multicategory (more precisely, a stable non-unital ∞-operad) with:
  - colours $F : \mathrm{Fun}_*^{\omega}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Sp}$;
  - multi-morphism spectra $(F_1, \dots, F_n) \to G$ given by

$$\mathrm{Nat}(G, F_1 \otimes \cdots \otimes F_n).$$

Proposition

Let $\mathbb{D}_{\mathcal{C}}$ be the full sub-∞-operad of $\mathrm{Fun}(\mathrm{Fun}_*^{\omega}(\mathcal{C}, \mathrm{Sp}), \mathrm{Sp})^{op,\otimes}$ whose colours are the objects of the form $\partial_1(-)(X; S^0)$ for $X \in \mathrm{Sp}(\mathcal{C})$. Then

$$\begin{aligned}\mathbb{D}_{\mathcal{C}}(X_1, \dots, X_n; Y) &\simeq \mathrm{Nat}(\partial_1(-)(Y), \partial_1(-)(X_1) \otimes \dots \partial_1(-)(X_n)) \\ &\simeq \partial_n I_{\mathcal{C}}(X_1, \dots, X_n; Y).\end{aligned}$$

Conjectures

- The constructions $\mathbb{D}_{\mathcal{C}}$ and $\mathbb{D}_F$ form a pseudofunctor between bicategories:
$$\mathbb{D} : \mathrm{Cat}_\infty^{*,\omega} \to \mathrm{Op}_\infty^{\mathsf{st,nu}}$$
- In particular, there is a chain rule:
$$\mathbb{D}_{GF} \simeq \mathbb{D}_G \circ_{\mathbb{D}_{\mathcal{D}}} \mathbb{D}_F.$$
- There is also a ∞-category of algebras pseudofunctor
$$\mathrm{Alg} : \mathrm{Op}_\infty^{\mathsf{st,nu}} \to \mathrm{Cat}_\infty^{*,\omega}$$
- Alg is the embedding of a reflective sub-bicategory, with right adjoint $\mathbb{D}$. In particular, we have
    - $\mathbb{D}\,\mathsf{Alg}(\mathcal{O}) \simeq \mathcal{O}$
    - $\mathcal{C} \to \mathsf{Alg}(\mathbb{D}\mathcal{C})$