Model bicategories and their homotopy bicategories

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From? LaBRI, Université de Bordeaux

When? CT 2018

[1] Joint with E. Dubuc and M. Szyld

Bicategorical Quillen's theorem

Theorem (Quillen)

C model category

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Overview

Before **model** bicategories

Cylinders and homotopies **with respect to a class of weak equivalences**

A localization of bicategories via homotopies

Model bicategories

Basic definitions

The homotopy bicategory

Main theorem

Final remarks and future work

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Setting $\mathcal{C}$ bicategory

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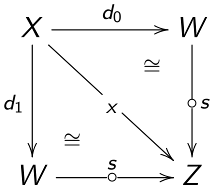

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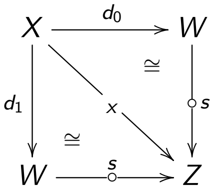

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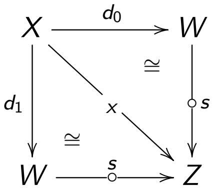

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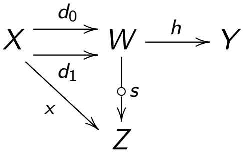

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$\mathcal{H}o(\mathcal{C}, \Sigma)$

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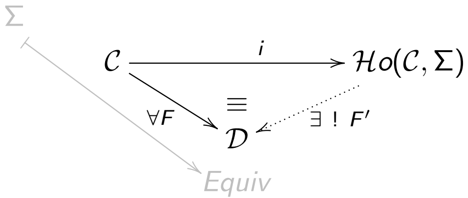

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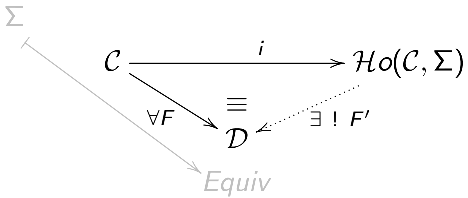

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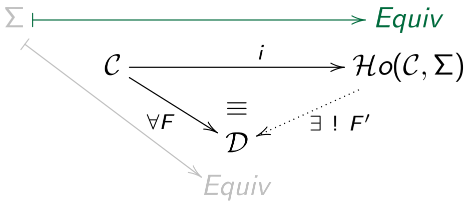

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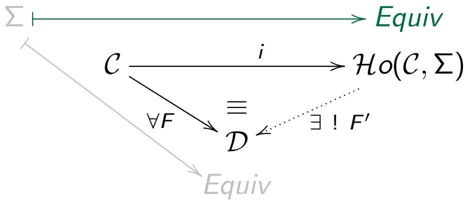

$\mathcal{H}o(\mathcal{C}, \Sigma)$ is the strong bicategorical localization at the weak equivalences.

Model bicategories

Definition of model bicategory

Three classes of morphisms, fibrations, cofibrations and weak equivalences, satisfying certain axioms (generalization of model category)

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not necessarily functorial

Quillen cylinders and homotopies

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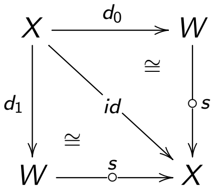

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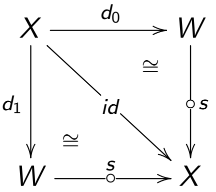

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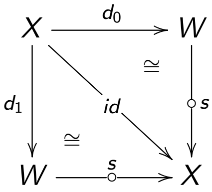

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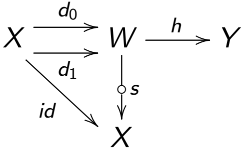

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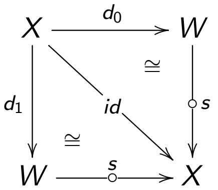

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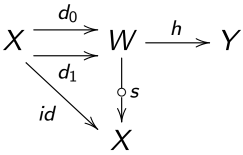

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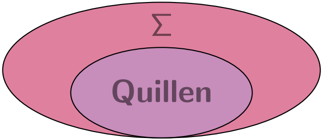

Homotopy bicategory

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$$\mathcal{H}o(\mathcal{C}) = \mathcal{H}o_{fc}(\mathcal{C}, \mathcal{W}) \xhookrightarrow{inc} \mathcal{H}o(\mathcal{C}, \mathcal{W})$$

Main theorem

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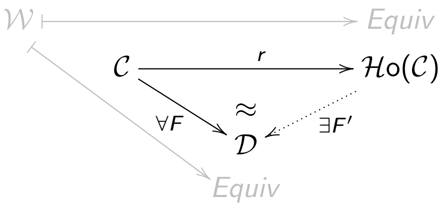

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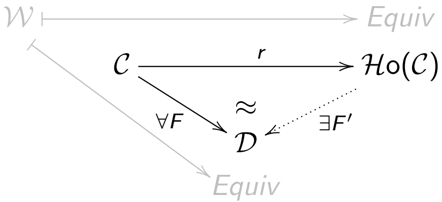

$$Hom(\mathcal{C}, \mathcal{H}o(\mathcal{C})) \xrightarrow{r^*} Hom_{\mathcal{W},Equiv}(\mathcal{C}, \mathcal{D})$$

is a pseudoequivalence of bicategories

Idea of the proof

$$\mathcal{C} \xrightarrow{\text{rep}} \mathcal{C}_{fc} \xhookrightarrow{inc} \mathcal{C} \xrightarrow{i} \mathcal{H}\text{o}(\mathcal{C})$$

$$r \equiv i \circ inc \circ \text{rep} : \mathcal{C} \to \mathcal{H}\text{o}(\mathcal{C})$$

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Step 3 r satisfies the universal property

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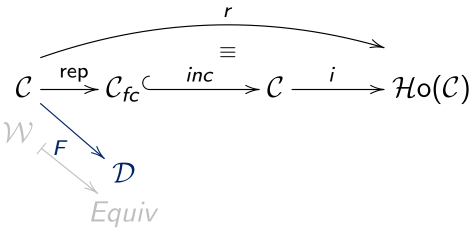

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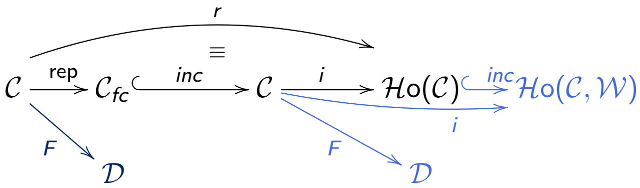

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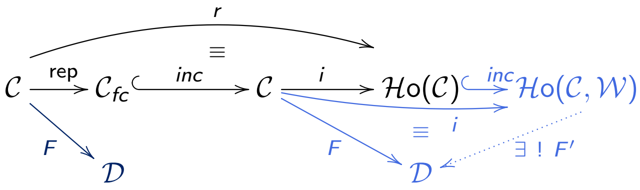

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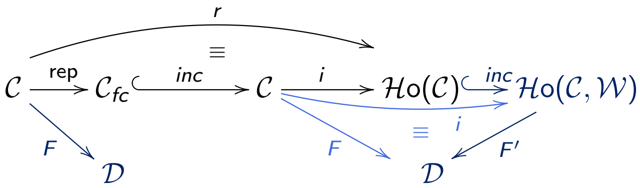

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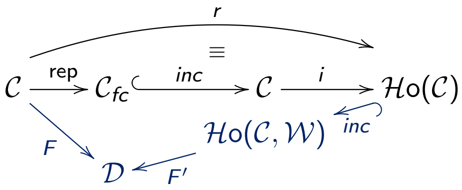

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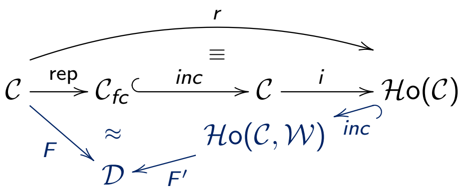

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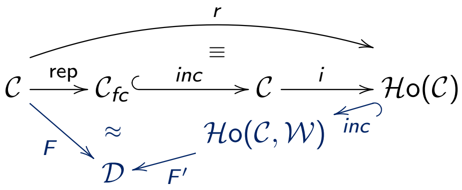

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$$rep(X) \underset{w.e.}{\longleftarrow} Y \underset{w.e.}{\longrightarrow} X$$

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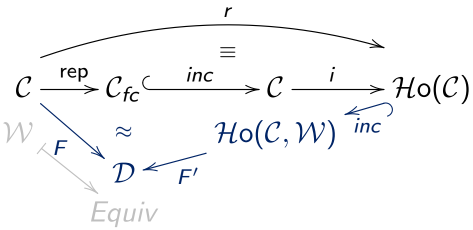

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$$rep(X) \xleftarrow[w.e.]{} Y \xrightarrow[w.e.]{} X$$

$$\rightsquigarrow$$

$$F' \circ inc \circ r(X) = F' \circ i \circ rep(X) = F \circ rep(X) \approx FX$$

Final remarks and future work

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Dimension 1	Recover Quillen's theorem
Applications	New invariants in strong shape theory
Examples	Other interesting examples of model bicategories

Thanks for your attention!