Zero, and some other ‘infinitesimal’ levels of a cohesive topos

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A quotation

The basic idea is simply to identify dimensions with levels and then try to determine what the general dimensions are in particular examples. More precisely, a space may be said to have (less than or equal to) the dimension grasped by a given level if it belongs to the negative (left adjoint inclusion) incarnation of that level. Thus a zero-dimensional space is just a discrete one (there are several answers, not gone into here, to the objection which general topologists may raise to that) and dimension one is the Aufhebung of dimension zero.

F. W. Lawvere
Some thoughts on the future of category theory
LNM 1488, 1991.

Axioms for the contrast of cohesion $\mathcal{E}$ and non-cohesion $\mathcal{S}$

Definition (Essentially in [L'07])

A geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive if the adjunction $p^* \dashv p_*$ extends to a string

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such that:

0. $p^* : \mathcal{S} \to \mathcal{E}$ is full and faithful,
1. (Nullstellensatz) the canonical $\theta : p_* \to p_!$ is epic and
2. $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products.

$$\texttt{pieces} \dashv \texttt{discr} \dashv \texttt{points} \dashv \texttt{codiscr}$$

Decidable objects

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If $\mathcal{S}$ is Boolean and $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive and locally connected then $p^* : \mathcal{S} \to \mathcal{E}$ coincides with $\mathbf{Dec}\mathcal{E} \to \mathcal{E}$.

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If Axiom 0 holds then the right adjoint $\mathcal{E} \to \mathbf{Dec}\mathcal{E}$ is the direct image of a hyperconnected geometric morphism (that we denote by $p : \mathcal{E} \to \mathbf{Dec}\mathcal{E}$).

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$\mathbf{Dec}\mathcal{E}$ is Boolean (well-known).
Then prove that $p_* : \mathcal{E} \to \mathbf{Dec}\mathcal{E}$ must coincide with $\neg\neg$-sheafification. $\square$

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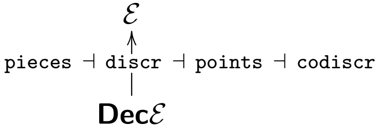

The UI of decidable objects and ¬¬-sheaves

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For details see:
The Unity and Identity of decidable objects and double negation sheaves.
To appear in the JSL.

Sufficient Cohesion, Quality types and Leibniz objects

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Proposition [L'07]

If p is both sufficiently cohesive and a quality type then $\mathcal{E} = 1 = \mathcal{S}$.

The canonical intensive quality

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive.

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The full subcategory $s^* : \mathcal{L} \to \mathcal{E}$ of Leibniz objects is the inverse image of a hyperconnected essential morphism $s : \mathcal{E} \to \mathcal{L}$ and, moreover,

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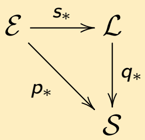

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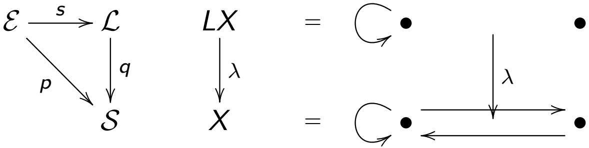

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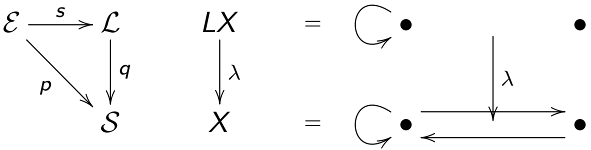


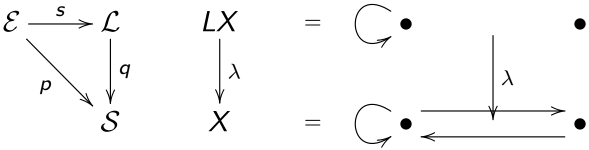
From [L'16]: "The Leibniz Core of a space X is the union $L(X)$ of all its generalized points; [...] The more general figures that substantiate cohesion between points are omitted in the reduction from X to $L(X)$, but each point may have self-cohesion (which is retained in $L(X)$)."

Birkhoff objects and how they relate with Birkhoff's Theorem

(Joint work with F. Marmolejo)

Motivated by Lawvere's paper
Birkhoff's Theorem from a geometric perspective: A simple example.
CGASA, 2016.

Birkhoff objects



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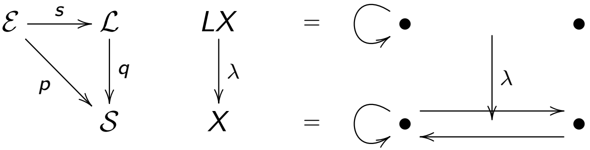

Definition

An object R in $\mathcal{E}$ is Birkhoff if every commutative diagram

$$LX \xrightarrow{\lambda} X \underset{g}{\overset{f}{\rightrightarrows}} R$$

implies $f = g$.

From [L'16]: "for any X, any 'infinitesimal' map $L(X) \to R$ can be integrated in at most one way to a global function $X \to R$."

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$A \xrightarrow{\text{monic}} \prod_{i \in I} S_i$	$(T_i \xrightarrow[\text{jointly epic}]{} X \mid i \in I)$	involves an induced 'pseudo-epimorphism' from an amalgam of special 'tiny' spaces.

A tentative ‘Birkhoff principle’

Algebra		Cohesion/Geometry
Hilbert’s Theorem $\rightsquigarrow$	epimorphic	$\texttt{points} \to \texttt{pieces}$
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Principle (for a pre-cohesive $p : \mathcal{E} \to \mathcal{S}$):

Birkhoff objects separate. (I.e. they form a separating class in $\mathcal{E}$.)

'There are enough Birkhoff objects'

The case of presheaf toposes 1 : Pseudo-constants

Let $\mathcal{C}$ be a category with 1.

Definition

A map $f : D \to C$ in $\mathcal{C}$ is a pseudo-constant if

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commutes for every $a, b : 1 \to D$.

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Representables are Birkhoff in $\widehat{\mathcal{C}}$ if and only if for every C in $\mathcal{C}$, the family of pseudo-constants with codomain C is jointly epic in $\mathcal{C}$ $\square$

SITE

$$\underbrace{(T_i \xrightarrow{i} X \mid i \text{ pseudo-constant})}_{\text{jointly epic}}$$

GEOMETRY/Cohesion

Birkhoff objects separate (B-principle)

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Birkhoff objects separate in the classifier on non-trivial BA's, that of 'connected' dLatt's , simplicial sets , reflexive graphs.

As in the case of reflexive graphs studied in [L'16],in all these examples Birkhoff objects coincide with $\neg\neg$-separated objects.

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Birkhoff objects separate in the classifier of $\mathbb{C}$-algebras without idempotents (as a pre-cohesive topos over **Set**).

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Note: In this case, Birkhoff does not imply $\neg\neg$-separated.

‘Infinitesimal’ levels and Birkhoff objects

The consideration of Birkhoff objects leads to the consideration of ‘infinitesimal’ subtoposes.

‘Infinitesimal’ subtoposes

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive.

Definition

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This happens in all the examples we mentioned.
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A more definite existence result

Let $\mathcal{C}$ be small, with 1, and s.t. every object has a point so that $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive.

Proposition

If every pseudo-constant in $\mathcal{C}$ factors through an object that has exactly one point then p has a level ϵ.

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For example, $1 \to \Delta$.

More interestingly, for the Gaeta topos of $\mathbb{C}$, the objects of ${\mathcal{C}_0}^{op}$ are the finite dimensional local $\mathbb{C}$-algebras.

‘Infinitesimal’ levels are below 1; as they should.

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive.

Proposition

If a subquality $\mathcal{F} \to \mathcal{E}$ is way-above level 0 then $\mathcal{S}$ is degenerate.

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Using the characterization of levels way-above 0 in M. Roy's thesis. □

Another quotation from L's thoughts on the future of CT

The infinitesimal spaces, which contain the base topos in its non-Becoming aspect, are a crucial step toward determinate Becoming, but fall short of having among themselves enough connected objects, i.e. they do not in themselves constitute fully a 'category of cohesive unifying Being.' In examples the four adjoint functors relating their topos to the base topos coalesce into two (by the theorem that a finite-dimensional local algebra has a unique section of its residue field) and the infinitesimal spaces may well negate the largest essential subtopos of the ambient one which has that property. This level may be called 'dimension ϵ'

Bibliography I

G. Birkhoff.
Subdirect unions in Universal Algebra.
BAMS, 1944.

F. W. Lawvere.
Birkhoff's Theorem from a geometric perspective: A simple example.
CGASA, 2016.

N. H. McCoy.
Subdirectly irreducible commutative rings.
Duke Math. J., 1945.

J. Emsalem.
Géométrie des points épais.
BSMF, 1978.

Bibliography II

A. Carboni and G. Janelidze.
Decidable (=separable) objects and morphisms in lextensive categories.
JPAA, 1996.

F. W. Lawvere.
Axiomatic Cohesion.
TAC, 2007.

M. Roy.
The topos of Ball Complexes.
Ph.D. thesis, University of New York at Buffalo, 1997.

W. Tholen
Nullstellen and Subdirect Representation
ACS, 2014.