Effective computations in predicative mathematics

Category Theory 2018

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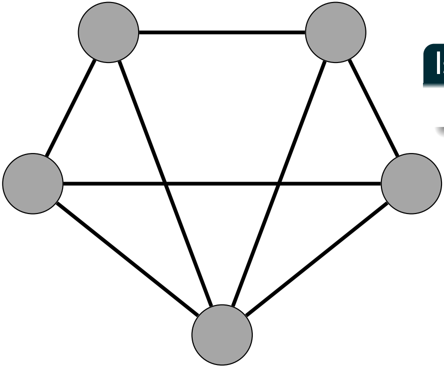
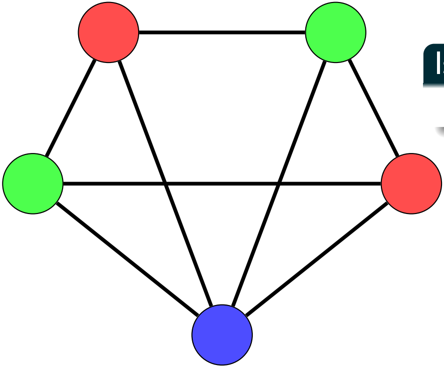


Is this graph 3-colorable?



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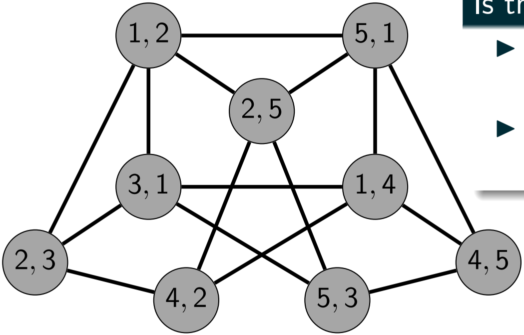
- Yes!



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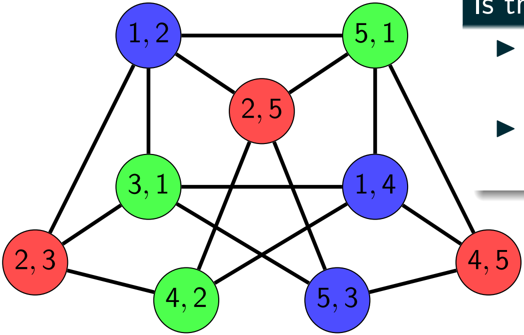
- Nodes: pairs of distinct natural numbers
- Edges: $n, m \leftrightarrow m, k$ whenever $n \neq k$

Algorithms
3-colorability



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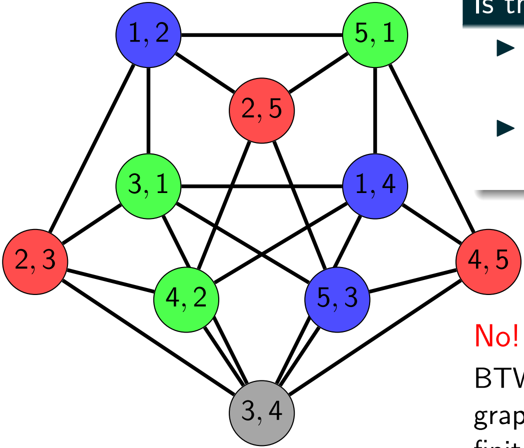

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No!

BTW, by compactness of FOL, a graph is 3-colorable iff its every finite subgraph is 3-colorable.

Puzzle
Linear equations

- Consider the following set of linear equations[1]:

$$x_{m,n} + x_{n,k} + x_{k,m} = 0$$
$$x_{0,1} + x_{1,0} = 1$$

for pairwise distinct natural numbers m, n, k

- Does this set of equations have a solution in $\mathcal{Z}_2$?
- If you cannot answer, you may write a program that solves the puzzle :-)

Sets with atoms
Cumulative hierarchy

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  - $V_0(A) = A$
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  - $\{\{0, 1\}, \{6, 7, 8\}, 7\} \mapsto \{\{1, 2\}, \{6, 7, 8\}, 6\}$

- (Remember: think of algebraic structure A as the set of natural numbers $\mathcal{N}$ with equality $=$.), define:
  - set-wise stabiliser of $X \in V$ in $Aut(A)$ as $Aut(A)_X = \{h \in Aut(A) \colon h \bullet X = X\}$
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Sets with atoms
Support

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- $X \in V$ is of finite support if there exists a finite $S \subseteq A$ that supports X
- $X \in V$ is legitimate if it is hereditarily of finite support
- We shall restrict to legitimate sets only.
- $X \in V$ is equivariant if it is supported by the empty set
- $X \in V$ is coherent if it has only finitely many orbits

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- $\mathcal{N}^* = \{\langle\rangle, \langle 0\rangle, \langle 1\rangle, \dots, \langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 0\rangle, \dots, \langle 3, 7, 2\rangle, \dots\}$ is equivariant, but not coherent

- Generally, if X is coherent X^2 may be not :-(
- Example: $\langle \mathcal{Z}, + \rangle$:
  - $\mathcal{Z}$ has a single orbit — for every $x \leq y \in \mathcal{Z}$ there exists translation by $k = y - x$, which maps x to y
  - $\mathcal{Z}^2$ has infinitely many orbits — two pairs $\langle x, y \rangle, \langle x', y' \rangle$ are in the same orbit iff $y - x = y' - x'$

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Good structures
Roelcke precompact groups

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  - Theorem: A topological group G is coherent iff its classifying topos $\mathbf{Set}^{G^{op}}$ is coherent
  - BTW, for $G = Aut(A)$ this classifying topos is equivalent to the category of equivariant sets with atoms A :-)

- Therefore, every $Th(A)$-definable subset of A^k is a coherent sets with atoms
- Examples:
  - $N = \{\langle n, m \rangle \in \mathcal{N} \colon n \neq m\}$
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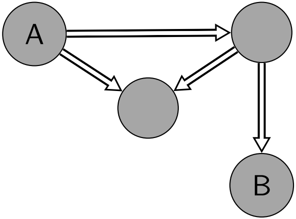
  - Fact: "nested" definable sets form the pretopos completion of definable sets



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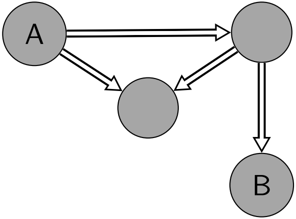

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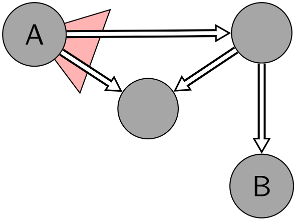

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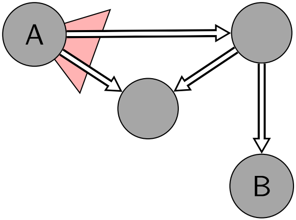

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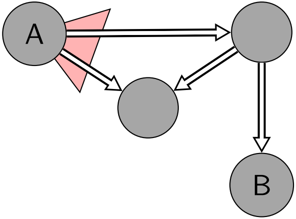
- (U/A)STCON on coherent graphs with atoms are decidable :-)



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$R \leftarrow \{A\}$
while $R' \neq R$ **do**
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  for $\langle x, y \rangle \in E$ **do**
    if $x \in R$ **then**
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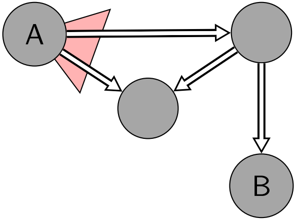
    end if
  end for
end while

Programming in sets with atoms
Reachability



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More exciting problems
Coherent Automata[4]

- Coherent alphabet Σ
- Coherent set Q of states
- Transition relation $\sigma \subseteq Q \times \Sigma \times Q$
- Initial state $q_0 \in Q$ and a coherent set of final states $F \subseteq Q$

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- $P \neq NP$ in sets with atoms
- Coherent automata over $\langle \mathcal{N}, = \rangle$ are equivalent to register automata

More exciting problems
Coherent Model Checking[5]

- A coherent μ-formula is given by the following syntax:

$$\phi ::= p \mid X \mid \bigvee \Phi \mid \neg\phi \mid \Diamond\phi \mid \mu X.\phi$$

where p is a proposition from an equivariant set $\mathcal{P}$, X is a variable, and Φ is a coherent set of μ-formulas.

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- Model checking for coherent μ-calculus over coherent Kripke structures is decidable :-)
- Model checking for coherent μ-calculus does not have a “finite/coherent-model” property

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- A coherent μ-formula is given by the following syntax:

$$\phi ::= p \mid X \mid \bigvee \Phi \mid \neg\phi \mid \Diamond\phi \mid \mu X.\phi$$

  where p is a proposition from an equivariant set $\mathcal{P}$, X is a variable, and Φ is a coherent set of μ-formulas.
- A Kripke structure consists of a set of states K, a transition relation $R \subseteq K \times K$, and an interpretation $W \subseteq \mathcal{P} \times K$
- Satisfiability of coherent μ-calculus is undecidable :-(
- Model checking for coherent μ-calculus over coherent Kripke structures is decidable :-)
- Model checking for coherent μ-calculus does not have a “finite/coherent-model” property
- $\exists(\bigwedge_{a \in A} G(p_a \rightarrow X(G\neg p_a)))$ is not expressible in coherent μ-calculus, but its model-checking is decidable

More exciting problems
Constraint satisfaction problem[1]

- A purely relational structure T is called template
- An instance I of template T is a structure in the language of T
- A solution of I over T is a homomorphism $s\colon I \to T$
- Example (3-colorability): given a graph $\langle V, E\rangle$ define: $T = \langle \{R, G, B\}, \neq \rangle$, $I = \langle V, E\rangle$

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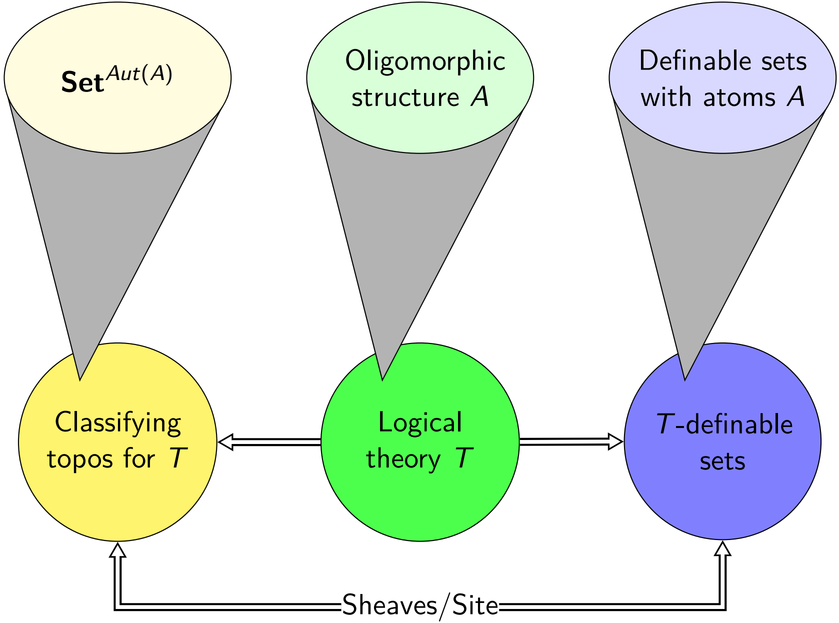
- Let T be a finite template such that CSP(T) is complete for a complexity class C under logarithmic space reductions. Then CSP-Inf(T) is decidable and complete for the complexity class exp(C) under logarithmic space reductions.
- For any equivariant, locally finite template, it is decidable whether a given definable, equivariant instance over it has a solution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Oligomorphic structure A
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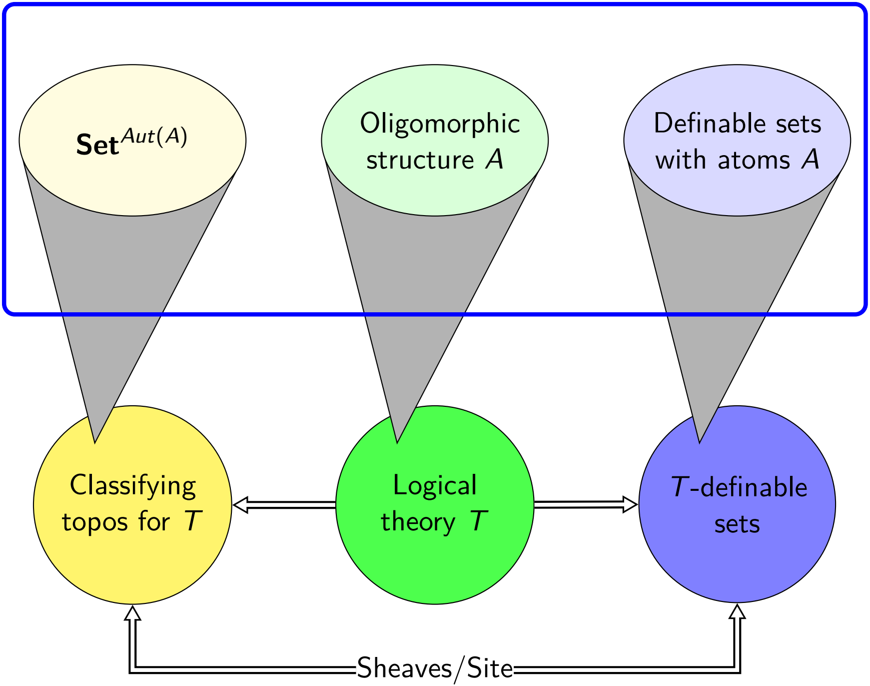
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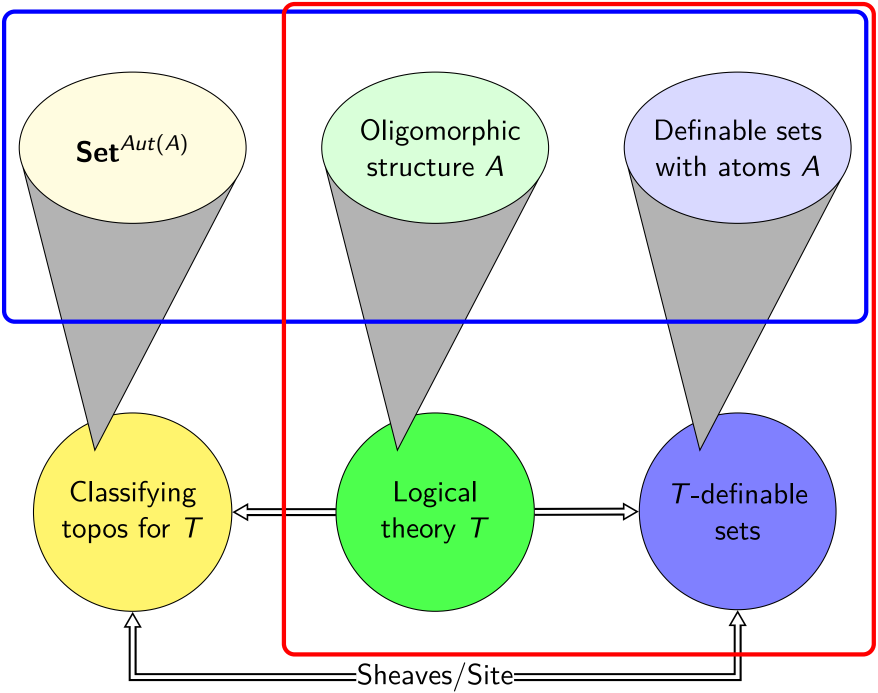

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Definable sets
Algorithms

- Fix a decidable FO theory T, such that every finite set of formulas generates a finite set (under logical operations)
- Let $\mathcal{G} = (N, E)$ be a T-definable graph
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- Let $\mathcal{G} = (N, E)$ be a T-definable graph
- Is the reachability problem for $\mathcal{G}$ decidable? — Yes!
- Assume that nodes N are represented by formula ψ, and edges E are represented by formula ϕ.

comment: $T' \subseteq T$ store consecutive approximations to t.c. of ϕ
$T' \leftarrow \emptyset$
$T \leftarrow \{\langle \overline{x}, \overline{x} \rangle : \psi(\overline{x})\}$
while $T' \neq T$ **do**
  $T' \leftarrow T$
  $T \leftarrow T \cup \{\langle \overline{x}, \overline{y} \rangle : \exists_{\overline{z}} \langle \overline{x}, \overline{z} \rangle \in T \wedge \phi(\overline{z}, \overline{y})\}$
end while

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  - There must be a well-defined notion of union of subobjects
- Fact: A category with finite limits, existential quantifiers and well-behaved unions is just a coherent category :-)

Beyond classifying toposes

- Closure properties:
  - products and cofiltered limits of coherent groups are coherent
  - (finite) products of coherent toposes are coherent toposes
  - products and filtered colimits of pretoposes are pretoposes
- Most of the results survive when moving to the filtered colimits of classifying toposes

Further Reading I

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