

# Conjugation semigroups and conjugation monoids with cancellation

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CT 2018  
Category Theory 2018

July 08-14, Ponta Delgada, Azores

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A conjugation semigroup  $(S, +, \overline{\phantom{x}})$  is a semigroup  $(S, +)$  equipped with a unary operation  $\overline{\phantom{x}} : S \rightarrow S$  satisfying the following identities:

- 1  $\overline{x} + x = x + \overline{x}$
- 2  $x + \overline{y} + y = y + \overline{y} + x$
- 3  $\overline{(x + y)} = \overline{y} + \overline{x}$

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## Examples

- Any group with  $\overline{x} = x^{-1}$ .
- Any commutative monoid with  $\overline{x} = e$ .

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## Examples

- Any group with  $\overline{x} = x^{-1}$ .
- Any commutative monoid with  $\overline{x} = e$ .
- $S = \{q \in \mathbb{H} \mid 0 < \|q\| < 1\}$  with quaternion product and conjugation.

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The quasivariety  $\mathcal{S}$  of conjugation semigroups with *cancellation* is a weakly Mal'tsev category.

# Weakly Mal'tsev Category

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A finitely complete category is weakly Mal'tsev if for all pullbacks of split epimorphisms along split epimorphisms

$$\begin{array}{ccc} A \times B & C & \xleftarrow{e_2} C \\ \pi_1 \downarrow & \uparrow e_1 & \xrightarrow{\pi_2} \\ A & & B \\ \uparrow f & \xleftarrow{r} & \downarrow g \\ & & s \end{array}$$

the pair  $(e_1, e_2)$ , with  $e_1 = \langle 1_A, sf \rangle$  and  $e_2 = \langle rg, 1_C \rangle$ , is jointly epimorphic.



# Weakly Mal'tsev Category

Examples of weakly Mal'tsev categories are

- DLat, property characterizing it amongst the varieties of lattices
- quasivarieties of algebras with a ternary operation  $p(x, y, z)$  satisfying

$$p(x, y, y) = p(y, y, x) \quad \text{and} \quad p(x, y, y) = p(x', y, y) \Rightarrow x = x'.$$

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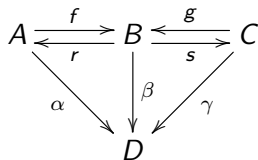
$$p(x, y, y) = p(y, y, x) \quad \text{and} \quad p(x, y, y) = p(x', y, y) \Rightarrow x = x'.$$

In  $\mathcal{S}$  we have

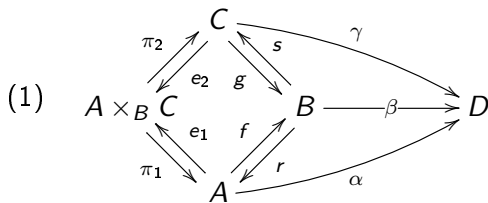
$$p(x, y, z) = x + \bar{y} + z$$

# Admissibility diagrams

An admissibility diagram



gives rise to



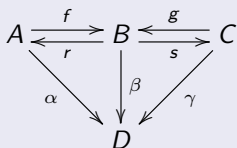
$$fr = 1_B = gs, \alpha r = \beta = \gamma s$$

The triple  $(\alpha, \beta, \gamma)$  is admissible with respect to  $(f, r, g, s)$  if there exists a unique morphism  $\varphi: A \times_B C \rightarrow D$  such that  $\varphi e_1 = \alpha$  and  $\varphi e_2 = \gamma$ .

Then we say that the diagram (1) is admissible.

## Theorem:

A diagram in  $\mathcal{S}$



$fr = 1_B = gs$ ,  $\alpha r = \beta = \gamma s$ , is admissible if and only if

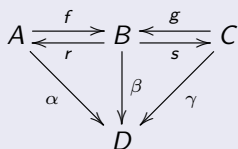
( $Ad_1$ ) the equation  $x + \overline{\beta(b)} + \beta(b) = \alpha(a) + \overline{\beta(b)} + \gamma(c)$  has a unique solution for all  $a \in A$  and  $c \in C$  such that  $f(a) = g(c) = b \in B$ .

( $Ad_2$ ) the equation

$$\alpha(a_1 + a_2) + \overline{\beta(b_1 + b_2)} + \gamma(c_1 + c_2) = \alpha(a_1) + \overline{\beta(b_1)} + \gamma(c_1) + \alpha(a_2) + \overline{\beta(b_2)} + \gamma(c_2)$$

is satisfied for  $a_1, a_2 \in A$  and  $c_1, c_2 \in C$  such that  $f(a_1) = g(c_1) = b_1 \in B$  and  $f(a_2) = g(c_2) = b_2 \in B$ .

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Also valid in  $\mathcal{M}$ , the category of conjugation monoids with cancellation. .

## Sketch of proof:

Existence of a map  $\varphi : A \times_B C \rightarrow D$  with  $\varphi e_1 = \alpha$  and  $\varphi e_2 = \gamma$  implies that, for  $f(a) = g(c) = b$ ,

$$\alpha(a) = \varphi(a, s(b)), \quad \gamma(c) = \varphi(r(b), c), \quad \beta(b) = \varphi(r(b), s(b)).$$

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$\varphi \in \mathcal{S} \Rightarrow \varphi(a, c)$  is the solution of

$$x + \overline{\beta(b)} + \beta(b) = \alpha(a) + \overline{\beta(b)} + \gamma(c)$$

and  $(Ad_2)$  is fulfilled.

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If  $(Ad_1)$  and  $(Ad_2)$  hold, taking  $\varphi(a, c)$  the solution of  $(Ad_1)$  then  $\varphi e_1 = \alpha$  and  $\varphi e_2 = \gamma$  and  $\varphi \in \mathcal{S}$ .



# Schreier split epimorphisms of monoids

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In  $Mon$

$$X \xrightarrow{k} A \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{f} \end{array} B \quad \text{with } fr = 1_B \text{ and } X = \ker f$$

is a Schreier split epi if there exists a *unique* set-theoretical map  $q : A \rightarrow X$ , called the Schreier retraction, such that  $a = kq(a) + rf(a)$  for all  $a \in A$ .

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To  $X \begin{array}{c} \xleftarrow{q} \\ \xrightarrow{k} \end{array} A \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{f} \end{array} B$  corresponds an action of  $B$  on  $X$ ,  $\varphi : B \rightarrow \text{End}(X)$

$$b \cdot x := \varphi(b)(x) = q(r(b) + k(x))$$

Conversely to each action  $\varphi : B \rightarrow \text{End}(X)$  it corresponds a Schreier split epimorphism via semidirect product.

# Schreier split epimorphism in $\mathcal{M}$

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Given a Schreier split epi in  $\mathcal{M}$

$$X \begin{array}{c} \xleftarrow{q} \\ \xrightarrow{k} \end{array} A \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{f} \end{array} B$$

we have:

- a)  $qk = 1_X$ ;
- b)  $qr = 0$ ;
- c)  $q(0) = 0$ ;
- d)  $k(b \cdot x) + r(b) = r(b) + k(x)$ ;
- e)  $q(a + a') = q(a) + q(rf(a) + q(a'))$ ;

$$f) \quad q(\bar{a}) = f(\bar{a}) \cdot \overline{q(a)}.$$

# Inducing internal structures

Given  $h : X \rightarrow B$  and a Schreier split epimorphism in  $\mathcal{M}$

$$\begin{array}{ccccc} & & h & & \\ & & \curvearrowright & & \\ X & \xleftarrow{q} & A & \xrightarrow{r} & B \\ & \xrightarrow{k} & & \xrightarrow{f} & \\ & & & & \end{array}$$

when does  $h$  induce:

- a reflexive graph,
- an internal category,
- an internal groupoid?

## Proposition

Given a Schreier split epimorphism and a morphism  $h$  in  $\mathcal{M}$

$$X \begin{array}{c} \xrightarrow{h} \\ \xleftarrow{q} \cdots \xrightarrow{k} A \xleftarrow{r} \xrightarrow{f} B \end{array},$$

$h$  induces a reflexive graph  $A \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{r} \\ \xrightarrow{\tilde{h}} \end{array} B,$

if and only if it satisfies the condition

$$(C_1) \quad h(b \cdot x) + b = b + h(x)$$

## Sketch of proof:

If there exists a map  $\tilde{h}$ , preserving addition and such that  $\tilde{h}k = h$  and  $\tilde{h}r = 1_B$ , then

$$\tilde{h}(a) = \tilde{h}(kq(a) + rf(a)) = hq(a) + f(a),$$

from which it follows that  $\overline{\tilde{h}(a)} = f(\bar{a}) + \overline{hq(a)}$

and so

$$\tilde{h}(\bar{a}) = \overline{\tilde{h}(a)}.$$



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and so

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The existence of such  $\tilde{h}$  is equivalent to  $(C_1)$ .

## Proposition

Given a Schreier split epi and a morphism  $h$  in  $\mathcal{M}$

$$\begin{array}{ccccc}
 & & h & & \\
 & \curvearrowright & & \curvearrowleft & \\
 X & \xleftarrow{\dots q \dots} & A & \xleftarrow{r} & B \\
 & \xrightarrow{k} & & \xrightarrow{f} & \\
 & & & & 
 \end{array}$$

$h$  induces an internal category

$$A \times_B A \xrightarrow{m} A \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{r} \\ \xrightarrow{\tilde{h}} \end{array} B$$

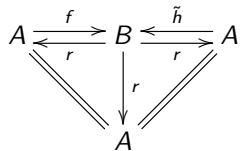
if and only if

$$(C_1) \quad h(b \cdot x) + b = b + h(x), \quad \forall x \in X, \forall b \in B$$

$$(C_2) \quad h(y) \cdot x + y = y + x, \quad \forall x, y \in X.$$

# Sketch of proof:

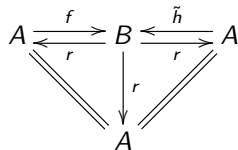
The reflexive graph  $A \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{r} \\ \xrightarrow{\tilde{h}} \end{array} B$  is an internal category if and only if the diagram



is admissible.

## Sketch of proof:

The reflexive graph  $A \begin{matrix} \xrightarrow{f} \\ \xleftarrow{r} \\ \xrightarrow{\tilde{h}} \end{matrix} B$  is an internal category if and only if the diagram



is admissible.

Then if  $(C_2)$  holds, such an  $m : A \times_B A \rightarrow A$  defining a Schreier internal category  $A \times_B A \xrightarrow{m} A \begin{matrix} \xrightarrow{f} \\ \xleftarrow{r} \\ \xrightarrow{\tilde{h}} \end{matrix} B$  exists, and is defined by

$$m(a, a') = kq(a) + a'$$

And  $(C_2)$  is also a necessary condition.

## Proposition

Given a Schreier split epimorphism and a morphism  $h$  in  $\mathcal{M}$

$$X \begin{array}{c} \xrightarrow{h} \\ \xleftarrow{q} \cdots \xrightarrow{k} A \xleftarrow{r} \xrightarrow{f} B \end{array}, \quad h \text{ induces an internal groupoid}$$

$$A \times_B A \xrightarrow{m} A \begin{array}{c} \xrightarrow{t} \\ \xleftarrow{r} \xrightarrow{f} B \\ \xrightarrow{\tilde{h}} \end{array}$$

if and only if

$$(C_1) \quad h(b \cdot x) + b = b + h(x), \quad \forall x \in X, \forall b \in B$$

$$(C_2) \quad h(y) \cdot x + y = y + x, \quad \forall x, y \in X.$$

$$(C_3) \quad X \text{ is a group and } -\bar{x} = \overline{-x}$$

# Sketch of proof:

The internal category

$$A \times_B A \xrightarrow{m} A \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{r} \\ \xrightarrow{\tilde{h}} \end{array} B$$

is an internal groupoid with the inverses defined on the "object of morphism"  $A$  by

$$t(a) = -kq(a) + r\tilde{h}(a)$$

exactly when  $(C_3)$  is satisfied.

# Example

$$B = \{q \in \mathbb{H} : \|q\| = 1\}$$

$$X = \{q \in \mathbb{H} : 0 < \|q\| \leq 1\}$$

$$b \cdot x = bxb^{-1} = bx\bar{b}$$

$$X \begin{array}{c} \xleftarrow{\pi_1} \\ \xrightarrow{\langle 1,0 \rangle} \end{array} X \times_{\varphi} B \begin{array}{c} \xleftarrow{\langle 0,1 \rangle} \\ \xrightarrow{\pi_2} \end{array} B$$

with  $\overline{(x, b)} = (\bar{b} \cdot \bar{x}, \bar{b})$  is a Schreier split epi in  $\mathcal{M}$ .

Given  $h : X \rightarrow B$ , such that  $h(x) = \frac{x}{\|x\|}$ ,  $h$  satisfies  $(C_1)$  (and so it induces a reflexive graph)

but not  $(C_2)$  (does not induce an internal category, in general).

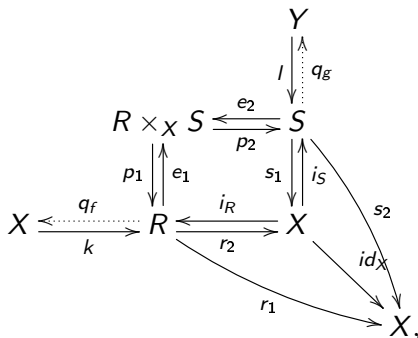
## Theorem

In the category  $\mathcal{M}$  of conjugation monoids with cancellation, two Schreier equivalence relations  $R$  and  $S$  on the same object  $X$  commute in the sense of Smith-Pedicchio if and only if their normalizations commute in the sense of Huq.



# "Smith is Huq"

Given two Schreier equivalence relations  $(R, r_1, r_2)$  and  $(S, s_1, s_2)$  on  $X$



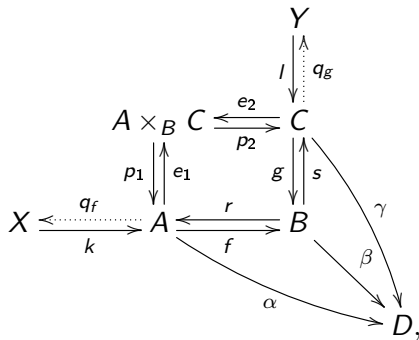
$r_1 k, s_2 l$  commute in Huq sense if and only if

$$\exists \varphi : R \times_X S \rightarrow X$$

such that  $\varphi e_1 = r_1$  and  $\varphi e_2 = s_2$ , and this means that  $R$  and  $S$  commute.

# From local to global

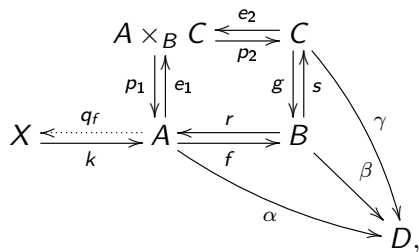
The diagram



is admissible if and only if  $\alpha k$  and  $\gamma l$  Huq-commute.

# From local to global

If just  $(f, r)$  is a Schreier split epi then the diagram

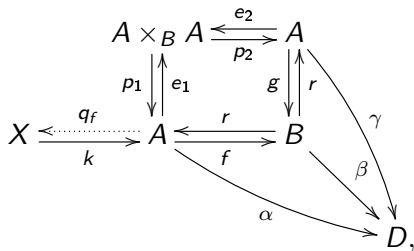


is admissible if and only if

$$\alpha k(q_f(c) \cdot x) + \gamma(c) = \gamma(c) + \alpha k(x) \text{ for all } x \in X \text{ and } c \in C.$$

# From local to global

If  $C = A$  and  $s = r$ , that is if we have a reflexive graph induced by  $h = gk$ , then the diagram



is admissible if and only if

$$\alpha k(h(y) \cdot x) + \gamma k(y) = \gamma k(y) + \alpha k(x), \text{ for all } x, y \in X.$$

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- [4] N. Martins-Ferreira and T. Van der Linden, A note on the “Smith is Huq” condition, *Appl. Categ. Structures* 20 (2012) 175–187.