Conjugation semigroups and conjugation monoids with cancellation

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Conjugation Semigroups

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A conjugation semigroup $(S, +, \overline{()})$ is a semigroup $(S, +)$ equipped with a unary operation $\overline{()} : S \to S$ satisfying the following identities:

1. $\overline{x} + x = x + \overline{x}$
2. $x + \overline{y} + y = y + \overline{y} + x$
3. $\overline{(x + y)} = \overline{y} + \overline{x}$

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- Any group with $\overline{x} = x^{-1}$.
- Any commutative monoid with $\overline{x} = e$.

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Examples

- Any group with $\overline{x} = x^{-1}$.
- Any commutative monoid with $\overline{x} = e$.
- $S = \{q \in \mathbb{H} | 0 < \|q\| < 1\}$ with quaternion product and conjugation.

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The quasivariety $\mathcal{S}$ of conjugation semigroups with *cancellation* is a weakly Mal'tsev category.

Weakly Mal'tsev Category

Weakly Mal'tsev Category

A finitely complete category is weakly Mal'tsev if for all pullbacks of split epimorphisms along split epimorphisms

$$\begin{tikzcd}
A \times_B C \arrow[r, shift right, "\pi_2"'] \arrow[d, shift right, "\pi_1"'] & C \arrow[l, shift right, "e_2"'] \arrow[d, shift right, "g"'] \\
A \arrow[r, shift right, "f"'] \arrow[u, shift right, "e_1"'] & B \arrow[l, shift right, "r"'] \arrow[u, shift right, "s"']
\end{tikzcd}$$

the pair (e_1, e_2), with $e_1 =< 1_A, sf >$ and $e_2 =< rg, 1_C >$, is jointly epimorphic.

Weakly Mal'tsev Category

Examples of weakly Mal'tsev categories are
- DLat, property characterizing it amongst the varieties of lattices
- quasivarieties of algebras with a ternary operation $p(x, y, z)$ satisfying

$$p(x, y, y) = p(y, y, x) \qquad \text{and} \qquad p(x, y, y) = p(x', y, y) \Rightarrow x = x'.$$

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In $\mathcal{S}$ we have

$$p(x, y, z) = x + \overline{y} + z$$

Admissibility diagrams

An admissibility diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[dr, "\alpha"'] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] \arrow[d, "\beta"] & C \arrow[l, shift right, "g"'] \arrow[dl, "\gamma"] \\
& D &
\end{tikzcd}$$

(1)

gives rise to

$$\begin{tikzcd}
& C \arrow[dl, shift left, "e_2"] \arrow[dr, shift left, "g"] \arrow[drr, bend left, "\gamma"] & & \\
A \times_B C \arrow[ur, shift left, "\pi_2"] \arrow[dr, shift left, "\pi_1"'] & & B \arrow[ul, shift left, "s"'] \arrow[dl, shift left, "r"] \arrow[r, "\beta"] & D \\
& A \arrow[ul, shift left, "e_1"'] \arrow[ur, shift left, "f"] \arrow[urr, bend right, "\alpha"'] & &
\end{tikzcd}$$

$fr = 1_B = gs$, $\alpha r = \beta = \gamma s$

The triple (α, β, γ) is admissible with respect to (f, r, g, s) if there exists a unique morphism $\varphi\colon A \times_B C \to D$ such that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$.

Then we say that the diagram (1) is admissible.

Admissibility in $\mathcal{S}$

Theorem:

A diagram in $\mathcal{S}$

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[dr, "\alpha"'] & B \arrow[l, shift left, "r"] \arrow[d, "\beta"] \arrow[r, shift right, "s"'] & C \arrow[l, shift right, "g"'] \arrow[dl, "\gamma"] \\
& D &
\end{tikzcd}$$

$fr = 1_B = gs$, $\alpha r = \beta = \gamma s$, is admissible if and only if

(Ad_1) the equation $x + \overline{\beta(b)} + \beta(b) = \alpha(a) + \overline{\beta(b)} + \gamma(c)$ has a unique solution for all $a \in A$ and $c \in C$ such that $f(a) = g(c) = b \in B$.

(Ad_2) the equation

$$\alpha(a_1+a_2)+\overline{\beta(b_1+b_2)}+\gamma(c_1+c_2) = \alpha(a_1)+\overline{\beta(b_1)}+\gamma(c_1)+\alpha(a_2)+\overline{\beta(b_2)}+\gamma(c_2)$$

is satisfied for $a_1, a_2 \in A$ and $c_1, c_2 \in C$ such that $f(a_1) = g(c_1) = b_1 \in B$ and $f(a_2) = g(c_2) = b_2 \in B$.

Admissibility in $\mathcal{S}$

Theorem:

A diagram in $\mathcal{S}$

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$$\alpha(a_1+a_2)+\overline{\beta(b_1+b_2)}+\gamma(c_1+c_2) = \alpha(a_1)+\overline{\beta(b_1)}+\gamma(c_1)+\alpha(a_2)+\overline{\beta(b_2)}+\gamma(c_2)$$

is satisfied for $a_1, a_2 \in A$ and $c_1, c_2 \in C$ such that $f(a_1) = g(c_1) = b_1 \in B$ and $f(a_2) = g(c_2) = b_2 \in B$.

Also valid in $\mathcal{M}$, the category of conjugation monoids with cancellation. .

Sketch of proof:

Existence of a map $\varphi : A \times_B C \to D$ with $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$ implies that, for $f(a) = g(c) = b$,

$$\alpha(a) = \varphi(a, s(b)),\ \gamma(c) = \varphi(r(b), c),\ \beta(b) = \varphi(r(b), s(b)).$$

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$\varphi \in \mathcal{S} \Rightarrow \varphi(a, c)$ is the solution of

$$x + \overline{\beta(b)} + \beta(b) = \alpha(a) + \overline{\beta(b)} + \gamma(c)$$

and (Ad_2) is fulfilled.

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and (Ad_2) is fulfilled.

If (Ad_1) and (Ad_2) hold, taking $\varphi(a, c)$ the solution of (Ad_1) then $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$ and $\varphi \in \mathcal{S}$.

Schreier split epimorphisms of monoids

Schreier split epimorphisms of monoids

In *Mon*

$$\begin{tikzcd} X \arrow[r, "k"'] & A \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd} \qquad \text{with } fr = 1_B \text{ and } X = kerf$$

is a Schreier split epi if there exists a *unique* set-theorical map $q : A \to X$, called the Schreier retraction, such that $a = kq(a) + rf(a)$ for all $a \in A$.

Schreier split epimorphisms of monoids

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To $\begin{tikzcd} X \arrow[r, shift right, "k"'] & A \arrow[l, shift right, dotted, "q"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd}$ corresponds an action of B on X, $\varphi : B \to End(X)$

$$b \cdot x := \varphi(b)(x) = q(r(b) + k(x))$$

Conversely to each action $\varphi : B \to End(X)$ it corresponds a Schreier split epimorphism via semidirect product.

Schreier split epimorphism in $\mathcal{M}$

Schreier split epimorphism in $\mathcal{M}$

Given a Schreier split epi in $\mathcal{M}$

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd}$$

we have:

- (a) $qk = 1_X$;
- (b) $qr = 0$;
- (c) $q(0) = 0$;
- (d) $k(b \cdot x) + r(b) = r(b) + k(x)$;
- (e) $q(a + a') = q(a) + q(rf(a) + q(a'))$;

- (f) $q(\bar{a}) = f(\bar{a}) \cdot \overline{q(a)}$.

Inducing internal structures

Given $h : X \to B$ and a Schreier spli epimorphism in $\mathcal{M}$

$$\begin{tikzcd} X \arrow[rr, bend left, "h"] \arrow[r, shift right, "k"'] & A \arrow[l, shift right, dotted, "q"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd}$$

when does h induce:

a reflexive graph,

an internal category,

an internal groupoid?

Inducing reflexive graphs

Proposition

Given a Schreier split epimorphism and a morphism h in $\mathcal{M}$

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] \arrow[rr, bend left, "h"] & A \arrow[l, dotted, shift right, "q"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd},$$

h induces a reflexive graph $\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r" description] \end{tikzcd}$,

if and only if it satisfies the condition

$$(C_1) \;\; h(b \cdot x) + b = b + h(x)$$

Sketch of proof:

If there exists a map $\tilde{h}$, preserving addition and such that $\tilde{h}k = h$ and $\tilde{h}r = 1_B$, then

$$\tilde{h}(a) = \tilde{h}(kq(a) + rf(a)) = hq(a) + f(a),$$

from which it follows that $\overline{\tilde{h}(a)} = f(\bar{a}) + h\overline{q(a)}$
and so

$$\tilde{h}(\bar{a}) = \overline{\tilde{h}(a)}.$$

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from which it follows that $\overline{\tilde{h}(a)} = f(\overline{a}) + h\overline{q(a)}$
and so

$$\tilde{h}(\overline{a}) = \overline{\tilde{h}(a)}.$$

The existence of such $\tilde{h}$ is equivalent to (C_1).

Inducing internal categories

Proposition

Given a Schreier split epi and a morphism h in $\mathcal{M}$

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] \arrow[rr, bend left, "h"] & A \arrow[l, shift right, dotted, "q"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd}$$

h induces an internal category

$$\begin{tikzcd} A \times_B A \arrow[r, "m"] & A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r"'] \end{tikzcd}$$

if and only if

$$(C_1)\ h(b \cdot x) + b = b + h(x),\ \forall x \in X, \forall b \in B$$

$$(C_2)\ h(y) \cdot x + y = y + x,\ \forall x, y \in X.$$

Sketch of proof:

The reflexive graph

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r" description] \end{tikzcd}$$

is an internal category if and only if the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[dr, equal] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "r"'] \arrow[d, "r"] & A \arrow[l, shift right, "\tilde{h}"'] \arrow[dl, equal] \\ & A & \end{tikzcd}$$

is admissible.

Sketch of proof:

The reflexive graph $\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r" description] \end{tikzcd}$ is an internal category if and only if the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[dr, equal] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "r"'] \arrow[d, "r"] & A \arrow[l, shift right, "\tilde{h}"'] \arrow[dl, equal] \\ & A & \end{tikzcd}$$

is admissible.

Then if (C_2) holds, such an $m : A \times_B A \to A$ defining a Schreier internal category $\begin{tikzcd} A \times_B A \arrow[r, "m"] & A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r" description] \end{tikzcd}$ exists, and is defined by

$$m(a, a') = kq(a) + a'$$

And (C_2) is also a necessary condition.

Inducing internal groupoids

Proposition

Given a Schreier split epimorphism and a morphism h in $\mathcal{M}$

$$\begin{tikzcd} X \arrow[r, "k"', shift right] \arrow[rr, bend left, "h"] & A \arrow[l, dotted, "q"', shift right] \arrow[r, "f"', shift right] & B \arrow[l, "r"', shift right] \end{tikzcd} , \qquad h \text{ induces an internal groupoid}$$

$$\begin{tikzcd} A \times_B A \arrow[r, "m"] & A \arrow[loop above, "t"] \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r"'] \end{tikzcd}$$

if and only if

$$(C_1)\ h(b \cdot x) + b = b + h(x),\ \forall x \in X, \forall b \in B$$

$$(C_2)\ h(y) \cdot x + y = y + x,\ \forall x, y \in X.$$

$$(C_3)\ X \text{ is a group and } -\overline{x} = \overline{(-x)}$$

Sketch of proof:

The internal category

$$\begin{tikzcd} A \times_B A \arrow[r, "m"] & A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r" description] \end{tikzcd}$$

is an internal groupoid with the inverses defined on the "object of morphism" A by

$$t(a) = -kq(a) + r\tilde{h}(a)$$

exactly when (C_3) is satisfied.

Example

$B = \{q \in \mathbb{H} : \|q\| = 1\}$
$X = \{q \in \mathbb{H} : 0 < \|q\| \leq 1\}$ $\qquad b \cdot x = bxb^{-1} = bx\overline{b}$

$$\begin{tikzcd}
X \arrow[r, shift right, "<1{,}0>"'] & X \times_{\varphi} B \arrow[l, dotted, shift right, "\pi_1"'] \arrow[r, shift right, "\pi_2"'] & B \arrow[l, shift right, "<0{,}1>"']
\end{tikzcd}$$

with $\overline{(x, b)} = (\overline{b} \cdot \overline{x}, \overline{b})$ is a Schreier split epi in $\mathcal{M}$.

Given $h : X \to B$, such that $h(x) = \frac{x}{\|x\|}$, h satisfies (C_1) (and so it induces a reflexive graph)
but not (C_2) (does not induce an internal category, in general).

"Smith is Huq"

Theorem

In the category $\mathcal{M}$ of conjugation monoids with cancellation, two Schreier equivalence relations R and S on the same object X commute in the sense of Smith-Pedicchio if and only if their normalizations commute in the sense of Huq.

"Smith is Huq"

Given two Schreier equivalence relations (R, r_1, r_2) and (S, s_1, s_2) on X

$$\begin{tikzcd}
 & & Y \arrow[d, shift right, "l"'] & \\
 & R \times_X S \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "p_1"'] & S \arrow[l, shift right, "e_2"'] \arrow[u, dotted, shift right, "q_g"'] \arrow[d, shift right, "s_1"'] \arrow[dr, bend left, "s_2"] & \\
X \arrow[r, shift right, "k"'] & R \arrow[l, dotted, shift right, "q_f"'] \arrow[u, shift right, "e_1"'] \arrow[r, shift right, "r_2"'] \arrow[drr, bend right, "r_1"'] & X \arrow[l, shift right, "i_R"'] \arrow[u, shift right, "i_S"'] \arrow[dr, "id_X"] & \\
 & & & X,
\end{tikzcd}$$

$r_1 k,\ s_2 l$ commute in Huq sense if and only if

$$\exists \varphi : R \times_X S \to X$$

such that $\varphi e_1 = r_1$ and $\varphi e_2 = s_2$, and this means that R and S commute.

From local to global

The diagram

$$\begin{tikzcd}
 & & Y \arrow[d, shift right, "l"'] & \\
 & A \times_B C \arrow[d, shift right, "p_1"'] \arrow[r, shift right, "p_2"'] & C \arrow[l, shift right, "e_2"'] \arrow[u, dotted, shift right, "q_g"'] \arrow[d, shift right, "g"'] \arrow[dr, bend left, "\gamma"] & \\
X \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q_f"'] \arrow[u, shift right, "e_1"'] \arrow[r, shift right, "f"'] \arrow[drr, bend right, "\alpha"'] & B \arrow[l, shift right, "r"'] \arrow[u, shift right, "s"'] \arrow[dr, "\beta"] & \\
 & & & D,
\end{tikzcd}$$

is admissible if and only if αk and γl Huq-commute.

From local to global

If just (f, r) is a Schreier split epi then the diagram

$$\begin{tikzcd}
 & A \times_B C \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "p_1"'] & C \arrow[l, shift right, "e_2"'] \arrow[d, shift right, "g"'] \arrow[ddr, bend left, "\gamma"] & \\
X \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q_f"'] \arrow[u, shift right, "e_1"'] \arrow[r, shift right, "f"'] \arrow[drr, bend right, "\alpha"'] & B \arrow[l, shift right, "r"'] \arrow[u, shift right, "s"'] \arrow[dr, "\beta"] & \\
 & & & D,
\end{tikzcd}$$

is admissible if and only if

$\alpha k(q_f(c) \cdot x) + \gamma(c) = \gamma(c) + \alpha k(x)$ for all $x \in X$ and $c \in C$.

From local to global

If $C = A$ and $s = r$, that is if we have a reflexive graph induced by $h = gk$, then the diagram

$$\begin{tikzcd}
 & A \times_B A \arrow[d, shift right, "p_1"'] \arrow[r, shift right, "p_2"'] & A \arrow[l, shift right, "e_2"'] \arrow[d, shift right, "g"'] \arrow[ddr, bend left, "\gamma"] & \\
X \arrow[r, shift right, "k"'] & A \arrow[u, shift right, "e_1"'] \arrow[l, dotted, shift right, "q_f"'] \arrow[r, shift right, "f"'] \arrow[drr, bend right, "\alpha"] & B \arrow[u, shift right, "r"'] \arrow[l, shift right, "r"'] \arrow[dr, "\beta"] & \\
 & & & D,
\end{tikzcd}$$

is admissible if and only if

$$\alpha k(h(y) \cdot x) + \gamma k(y) = \gamma k(y) + \alpha k(x), \text{ for all } x, y \in X.$$

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