Mal'tsev like categories

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A$\binom{z}{\text{ç}}$ores - Portugal

[1]

[1] Joint work with Z. Janelidze and T. Van der Linden

Anatoly Ivanovich Mal'tsev

Find a common setting to accommodate the three notions:

Mal'tsev category
naturally Mal'tsev category
weakly Mal'tsev category

Journal of Pure and Applied Algebra 69 (1990) 271–284
North-Holland

DIAGRAM CHASING IN MAL'CEV CATEGORIES

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Communicated by F.W. Lawvere
Received 8 February 1990

A diagram chasing technique generalizing the 'two-square' lemma of homological algebra is extended from Mal'cev varieties to Mal'cev categories: regular categories in which all reflexive relations are effective. The principal method used is the calculus of relations. The connection with Goursat's Theorem in group theory is discussed.

Naturally Mal'tsev category

Journal of Pure and Applied Algebra 61 (1989) 251-256
North-Holland

AFFINE CATEGORIES AND NATURALLY MAL'CEV CATEGORIES

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Communicated by P.J. Freyd
Received 9 August 1988

This note is a complement to a beautiful recent paper of A. Carboni, which characterizes affine categories, i.e. those categories which occur as slices of additive categories. We show that the condition that every reflexive graph has a unique groupoid structure, which was observed by Carboni to follow from affineness, is equivalent to the existence of a natural Mal'cev operation on a category; we further show that this condition implies the additiveness of the category of pointed objects, but not the affineness of the original category.

WEAKLY MAL'CEV CATEGORIES

N. MARTINS-FERREIRA

ABSTRACT. We introduce a notion of *weakly Mal'cev category*, and show that: (a) every internal reflexive graph in a weakly Mal'tsev category admits at most one multiplicative graph structure in the sense of [10] (see also [11]), and such a structure always makes it an internal category; (b) (unlike the special case of Mal'tsev categories) there are weakly Mal'tsev categories in which not every internal category is an internal groupoid. We also give a simplified characterization of internal groupoids among internal categories in this context.

1. Introduction

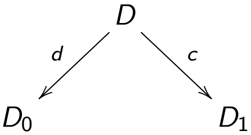
A weakly Mal'cev category (WMC) is defined by the following two axioms:

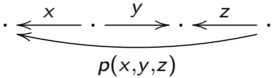
1. Existence of pullbacks of split epis along split epis.
2. Every induced canonical pair of morphisms into a pullback (see Definition 2.3), is jointly epimorphic.

Theory Appl. Categ. **21** (6) (2008) 91–117.

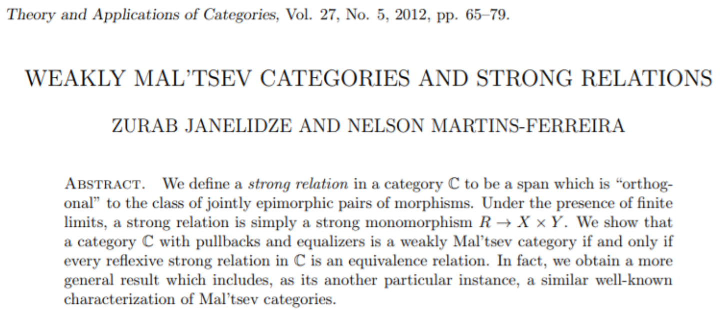
Let $\mathbb{C}$ be a category with pullbacks and equalizers. We will say that $\mathbb{C}$ is:

1. Mal'tsev, when every relation in C is difunctional
2. naturally Mal'tsev, when every span in C is uniquely equipped with a pregroupoid structure
3. weakly Mal'tsev, when every strong relation is difunctional





$d(x) = d(y), \quad c(y) = c(z) \qquad p(x,y,y) = x, \qquad p(y,y,z) = z$
$dp(x,y,z) = d(z), \quad cp(x,y,z) = c(x)$


Theory and Applications of Categories, Vol. 27, No. 5, 2012, pp. 65–79.

WEAKLY MAL'TSEV CATEGORIES AND STRONG RELATIONS

ZURAB JANELIDZE AND NELSON MARTINS-FERREIRA

ABSTRACT. We define a *strong relation* in a category $\mathbb{C}$ to be a span which is "orthogonal" to the class of jointly epimorphic pairs of morphisms. Under the presence of finite limits, a strong relation is simply a strong monomorphism $R \to X \times Y$. We show that a category $\mathbb{C}$ with pullbacks and equalizers is a weakly Mal'tsev category if and only if every reflexive strong relation in $\mathbb{C}$ is an equivalence relation. In fact, we obtain a more general result which includes, as its another particular instance, a similar well-known characterization of Mal'tsev categories.

Example - commutative magmas with cancellation

J. Fatelo and N. Martins-Ferreira, *Internal monoids and groups in the category of commutative cancellative medial magmas*, Portugaliae Mathematica **74** (3) (2016) 219–245.

Lemma 3.5. *Let $f, g\colon X \times Y \to B$ be any two morphisms in the category of ccm-magmas. Then:*

$$\begin{pmatrix} f(x,y) = g(x,y) \\ f(z,y) = g(z,y) \\ f(z,w) = g(z,w) \end{pmatrix} \Rightarrow f(x,w) = g(x,w).$$

8 J. P. Fatelo and N. Martins-Ferreira

Proof. Assuming $f(x,y) = g(x,y)$ and $f(z,w) = g(z,w)$ we have:

$$\begin{aligned} f(x,w) \oplus f(z,y) &= f(x \oplus z, w \oplus y) \\ &= f(x \oplus z, y \oplus w) \\ &= f(x,y) \oplus f(z,w) \\ &= g(x,y) \oplus g(z,w) \\ &= g(x \oplus z, y \oplus w) \\ &= g(x \oplus z, w \oplus y) \\ &= g(x,w) \oplus g(z,y). \end{aligned}$$

Now, if $f(z,y) = g(z,y)$ then we conclude that $f(x,w) = g(x,w)$ as desired, using axiom (M2). □

Distributive lattices

Lat category of lattices; **DLat** distributive lattices.

Theorem (NMF'12)

*Let $I\colon \mathbb{C} \to$ **Lat** be a full subcategory of lattices. TFAE:*

1. *every strong relation in $\mathbb{C}$ is difunctional;*
2. *I factors through* **DLat**

Further examples

$F\colon \mathbb{A} \to \mathbb{B}$ be a functor preserving finite limits. If there is a natural transformation $p_A\colon F(A) \times F(A) \times F(A) \to F(A)$ such that $p_A(x, y, y) = x = p_A(y, y, x)$, and

1. the functor is faithful, then $\mathbb{A}$ is a weakly Mal'tsev category;
2. the functor is faithful and conservative, then $\mathbb{A}$ is a Mal'tsev category [Pedicchio];
3. F is an isomorphism, then $\mathbb{A}$ is naturally Mal'tsev category [Johnstone]

First Unification Result

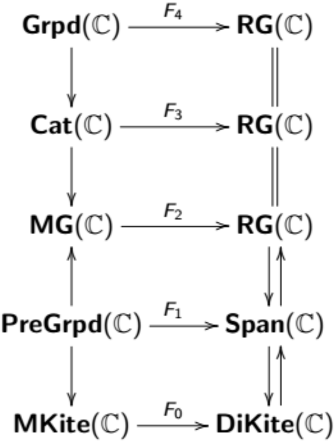
CATEGORIES VS. GROUPOIDS
VIA GENERALISED MAL'TSEV PROPERTIES

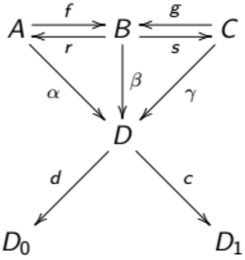
by Nelson MARTINS-FERREIRA and Tim VAN DER LINDEN

Résumé. On étudie la différence entre les catégories internes et les groupoïdes internes en termes de propriétés de Malcev généralisées—la propriété de Malcev faible d'un côté, et l'n-permutabilité de l'autre. Dans la première partie de l'article on donne des conditions sur les structures catégoriques internes qui détectent si la catégorie ambiante est naturellement de Malcev, de Malcev ou faiblement de Malcev. On démontre que celles-ci ne dépendent pas de l'existence de produits binaires. Dans la seconde partie on se concentre sur les variétés d'algèbres universelles.

Abstract. We study the difference between internal categories and internal groupoids in terms of generalised Mal'tsev properties—the weak Mal'tsev property on the one hand, and n-permutability on the other. In the first part of the article we give conditions on internal categorical structures which detect whether the surrounding category is naturally Mal'tsev, Mal'tsev or weakly Mal'tsev. We show that these do not depend on the existence of binary products. In the second part we focus on varieties of algebras.

Cah. Topol. Géom. Differ. Catég. LV (2014), no. 2, 83–112



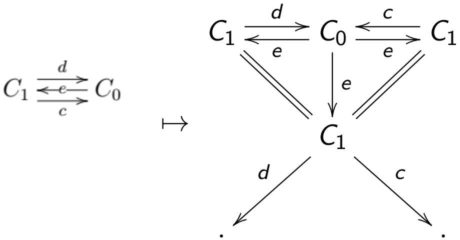


$$fr = 1_B = gs$$
$$\alpha r = \beta = \gamma s$$
$$d\alpha = d\beta f$$
$$c\beta g = c\gamma$$

A multiplication on a kite is a morphism $m \colon A \times_B C \to D$ such that $dm = d\gamma\pi_2$, $cm = c\alpha\pi_1$, $me_1 = \alpha$ and $me_2 = \gamma$.

Reflexive graph as a directed kite

If (C_1, C_0, d, e, c) is a reflexive graph then the following diagram is a directed kite

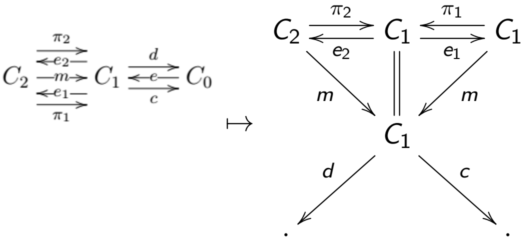


(1)

This directed kite is multiplicative if and only if the reflexive graph is a multiplicative graph.

Multiplicative graph as a directed kite

If (C_1, C_0, d, e, c, m) is a multiplicative graph then the following diagram is a directed kite

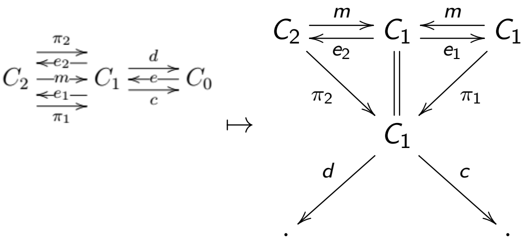


(2)

This directed kite has a unique multiplicative structure if and only if the multiplicative graph is associative (i.e., an internal category).

Directed kite derived from an internal category

If (C_1, C_0, d, e, c, m) is an associative multiplicative graph (that is, an internal category) then the following diagram is a directed kite



(3)

This directed kite is multiplicative if and only if the internal category is an internal groupoid.

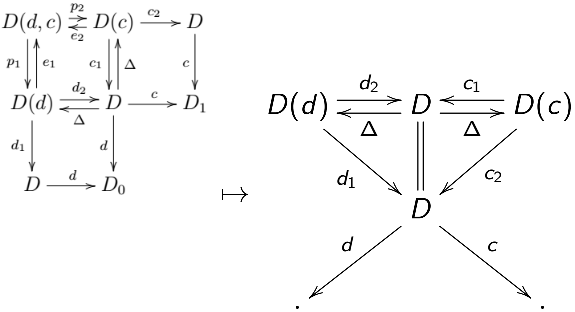
Morphism of reflexive graphs

If $(f_1, f_0)\colon (C_1, C_0, d, e, c) \to (C_1', C_0', d', e', c')$ is a morphism of reflexive graphs then the following diagram is a directed kite

$$
\begin{array}{ccc}
C_1 & \overset{d}{\underset{c}{\substack{\longrightarrow \\ \xleftarrow{e} \\ \longrightarrow}}} & C_0 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\
C_1' & \overset{d'}{\underset{c'}{\substack{\longrightarrow \\ \xleftarrow{e'} \\ \longrightarrow}}} & C_0'
\end{array}
\quad \mapsto \quad
\begin{array}{ccccc}
C_1 & \overset{d}{\underset{e}{\rightleftarrows}} & C_0 & \overset{c}{\underset{e}{\leftrightarrows}} & C_1 \\
& {\scriptstyle f_1}\searrow & \downarrow{\scriptstyle e' f_0} & \swarrow{\scriptstyle f_1} & \\
& & C_1' & & \\
& {\scriptstyle d'}\swarrow & & \searrow{\scriptstyle c'} & \\
\cdot & & & & \cdot
\end{array}
\qquad (4)
$$

The kernel pair construction as a directed kite

If (D, d, c) is a span then the kernel pair construction gives a directed kite as follows



(5)

This yields a reflection between the category of directed kites and the category of spans $\mathbf{DiKite} \rightleftarrows \mathbf{Span}$
A directed kite goes to its direction span, a span goes to the directed kite displayed above. Moreover, the span (D, d, c) is a pregroupoid if and only if its associated directed kite is multiplicative.

Theorem

Let $\mathbb{C}$ be a category with pullbacks and equalizers. If $\mathcal{M}$ is a class of spans in $\mathbb{C}$ which contains all identity spans and is stable under pullbacks, then t.f.c.a.e:

(a) $F_4 \colon \mathbf{Grpd}(\mathbb{C}, \mathcal{M}) \to \mathbf{RG}(\mathbb{C}, \mathcal{M})$ *has a section.*

(b) $F_3 \colon \mathbf{Cat}(\mathbb{C}, \mathcal{M}) \to \mathbf{RG}(\mathbb{C}, \mathcal{M})$ *has a section.*

(c) $F_2 \colon \mathbf{MG}(\mathbb{C}, \mathcal{M}) \to \mathbf{RG}(\mathbb{C}, \mathcal{M})$ *has a section.*

(d) $F_1 \colon \mathbf{PreGrpd}(\mathbb{C}, \mathcal{M}) \to \mathbf{Span}(\mathbb{C}, \mathcal{M})$ *has a section.*

(e) $F_4 \colon \mathbf{Grpd}(\mathbb{C}, \mathcal{M}) \to \mathbf{RG}(\mathbb{C}, \mathcal{M})$ *is an isomorphism.*

(f) $F_3 \colon \mathbf{Cat}(\mathbb{C}, \mathcal{M}) \to \mathbf{RG}(\mathbb{C}, \mathcal{M})$ *is an isomorphism.*

(g) $F_2 \colon \mathbf{MG}(\mathbb{C}, \mathcal{M}) \to \mathbf{RG}(\mathbb{C}, \mathcal{M})$ *is an isomorphism.*

(h) $F_1 \colon \mathbf{PreGrpd}(\mathbb{C}, \mathcal{M}) \to \mathbf{Span}(\mathbb{C}, \mathcal{M})$ *is an isomorphism.*

(h) $F_0 \colon \mathbf{MKite}(\mathbb{C}, \mathcal{M}) \to \mathbf{DiKite}(\mathbb{C}, \mathcal{M})$ *is an isomorphism.*

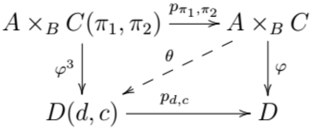
Sketch of proof

$$
\begin{array}{ccccc}
\mathbf{Grpd}(\mathbb{C},\mathcal{M}) & \xrightarrow{F_4^{\mathcal{M}}} & \mathbf{RG}(\mathbb{C},\mathcal{M}) & & \\
\downarrow & & \| & & \\
\mathbf{Cat}(\mathbb{C},\mathcal{M}) & \xrightarrow{F_3^{\mathcal{M}}} & \mathbf{RG}(\mathbb{C},\mathcal{M}) & & \\
\downarrow & & \| & & \\
\mathbf{MG}(\mathbb{C},\mathcal{M}) & \xrightarrow{F_2^{\mathcal{M}}} & \mathbf{RG}(\mathbb{C},\mathcal{M}) & & \\
\uparrow & & \downarrow\uparrow & & \\
\mathbf{PreGrpd}(\mathbb{C},\mathcal{M}) & \xrightarrow{F_1^{\mathcal{M}}} & \mathbf{Span}(\mathbb{C},\mathcal{M}) & \xleftarrow{\cong} & \mathcal{M} \\
\downarrow & & \downarrow\uparrow & & \\
\mathbf{MKite}(\mathbb{C},\mathcal{M}) & \xrightarrow{F_0^{\mathcal{M}}} & \mathbf{DiKite}(\mathbb{C},\mathcal{M}) & &
\end{array}
$$

Sketch of proof (cont.)

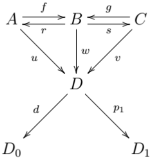
a section $\delta \colon A \times_B C \to A \times_B C(\pi_1, \pi_2)$, sending a pair (a, c) to the zigzag

$$a \longleftarrow sf(a) \longrightarrow rg(c) \longleftarrow c$$



from which we conclude $\varphi = p_{d,c}\theta$, since we have

$$\varphi = \varphi p_{\pi_1,\pi_2}\delta = p_{d,c}\varphi^3\delta = p_{d,c}\theta$$



Recap II - Uniform definition in terms of a class of spans

Let $\mathbb{C}$ be a category with pullbacks and equalizers. Suppose that $\mathcal{M}$ is a class of spans in $\mathbb{C}$ which contains all identity spans, is stable under pullbacks and the equivalent conditions of Theorem I are satisfied. We say that $\mathbb{C}$ is:

1. Mal'tsev, when the class $\mathcal{M}$ consists of all relations (monic spans)
2. naturally Mal'tsev, when the class $\mathcal{M}$ consists of all spans
3. weakly Mal'tsev, when the class $\mathcal{M}$ consists of all strong relations (strongly monic spans)

`This is very good!, but not yet good enough...`

Fibration of pointed objects

Mal'cev Categories and Fibration of Pointed Objects

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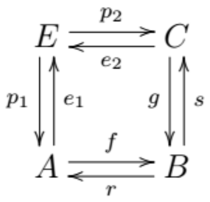
(Received: 5 December 1994; accepted: 6 July 1995)

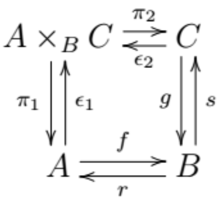
Abstract. The fibration p of pointed objects of a category **E** is shown to have some classifying properties: it is additive if and only if **E** is naturally Mal'cev, it is unital if and only if **E** is Mal'cev. The category **E** is protomodular if and only if the change of base functors relative to p are conservative.

Appl. Categ. Struct. **4** (2–3) (1996) 307–327

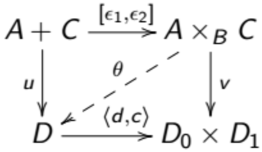
Orthogonality between spans and cospans



split square



split pullback



cospan $\perp$ span

Works well, but only when spans are monic (relations)

Moving to arbitrary spans required the discovery of a new concept: instead of a cospan being orthogonal to a span, we have to consider a split square compatible with a span.

Compatibility between a split square and a span

Definition

A split square, such as (previous slide), is said to be compatible with a span (D, d, c) if for every morphism

$$u \colon E \to D$$

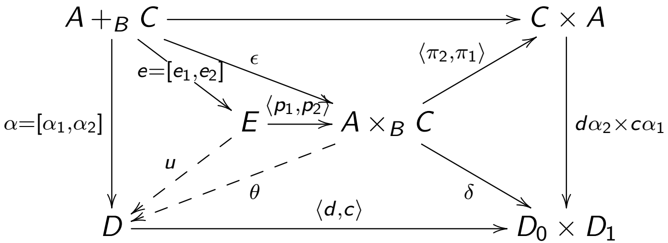
with $du = due_2p_2$ and $cu = cue_1p_1$, there exists a unique morphism

$$\theta \colon A \times_B C \to D$$

such that $\theta\epsilon_1 = ue_1$, $\theta\epsilon_2 = ue_2$, $d\theta = due_2\pi_2$ and $c\theta = cue_1\pi_1$.

When a split square is compatible with all the spans from a class of spans $\mathcal{M}$ then we say that it is $\mathcal{M}$-compatible.

In the presence of products and pushouts:



For every outer commutative diagram, if there exists $u\colon E \to D$ such that $ue = \alpha$ and $\langle d, c\rangle u = \delta\langle p_1, p_2\rangle$, then there exists a unique morphism θ such that $\theta\epsilon = \alpha$ and $\langle d, c\rangle\theta = \delta$.

Theorem

Let $\mathbb{C}$ be a category with pullbacks and equalizers. If $\mathcal{M}$ is a class of spans in $\mathbb{C}$ which contains all identity spans and is stable under pullbacks, then t.f.c.a.e:

(a) $F_4 \colon \mathbf{Grpd}(\mathbb{C},\mathcal{M}) \to \mathbf{RG}(\mathbb{C},\mathcal{M})$ *has a section.*

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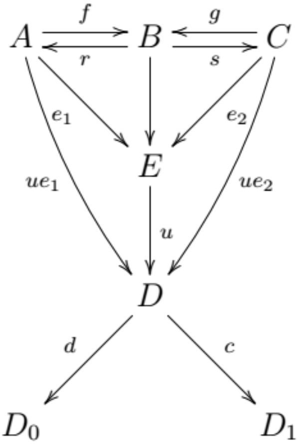
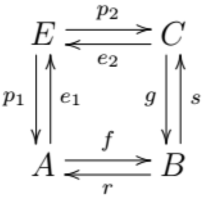
(g) $F_2 \colon \mathbf{MG}(\mathbb{C},\mathcal{M}) \to \mathbf{RG}(\mathbb{C},\mathcal{M})$ *is an isomorphism.*

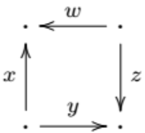
(h) $F_1 \colon \mathbf{PreGrpd}(\mathbb{C},\mathcal{M}) \to \mathbf{Span}(\mathbb{C},\mathcal{M})$ *is an isomorphism.*

(i) *Every split square in $\mathbb{C}$ is $\mathcal{M}$-compatible.*

Sketch of proof







Many Thanks and

Thank You