Towards a type theory for directed homotopy theory

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Outline

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Goal

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To develop a directed type theory.

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To formalize theorems about:

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- Directed paths are introduced as terms of a type former, hom, to be added to Martin-Löf type theory

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- Transport along terms of hom

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Criteria

- Directed paths are introduced as terms of a type former, hom, to be added to Martin-Löf type theory
- Transport along terms of hom
- Independence of hom and Id

What does directed mean?

Syntactically

Martin-Löf's identity type is symmetric/undirected since for any type T, and terms $a, b : T$, there is a function

$$i : \mathrm{Id}_T(a, b) \to \mathrm{Id}_T(b, a)$$

so that any *path* $p : \mathrm{Id}_T(a, b)$ can be *inverted* to obtain a path $ip : \mathrm{Id}_T(b, a)$.

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- Can think of these terms as *undirected* paths
- Can we design a type former of *directed* paths that resembles Id but without its inversion operation i?

What does directed mean?

Theorem

$\mathcal{C}$ cartesian closed category. A functorial reflexive relation

$$1_{\mathcal{C}} \xrightarrow{r} Id \xrightarrow{\epsilon_0 \times \epsilon_1} 1_{\mathcal{C}} \times 1_{\mathcal{C}}$$

models identity types if and only if the mapping path space factorization

$$X \xrightarrow{f} Y \quad \rightsquigarrow \quad X \xrightarrow{1 \times rf} X \times_Y Id(Y) \xrightarrow{\epsilon_1} Y$$

generates a weak factorization system on $\mathcal{C}$ where all red (resp. blue) maps are in the left (resp. right) class.

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models identity types if and only if it is

1. transitive,
2. homotopical,
3. symmetric.

What does directed mean?

Semantically

higher groupoids

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(undirected paths $\subseteq$
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Rough definition

A space together with a subset of its paths that are marked as 'directed'

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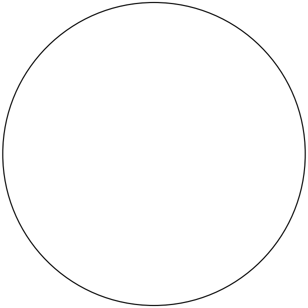

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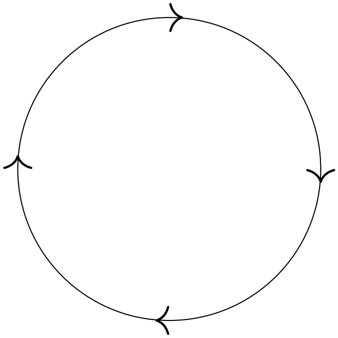

Application: concurrency

Concurrent processes can be represented by directed spaces.

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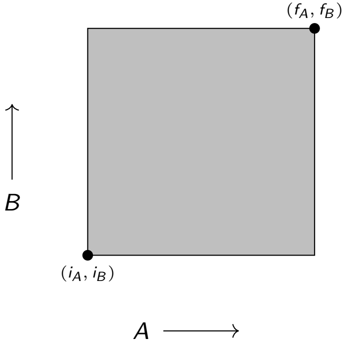

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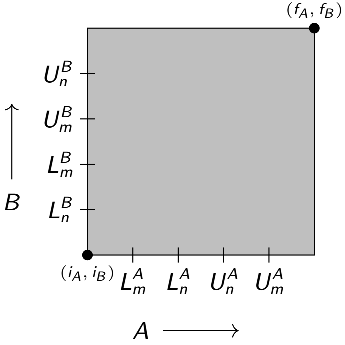

- A, B are two processes
- m, n are two memory locations
- which can be locked (L) or unlocked (U) by each process

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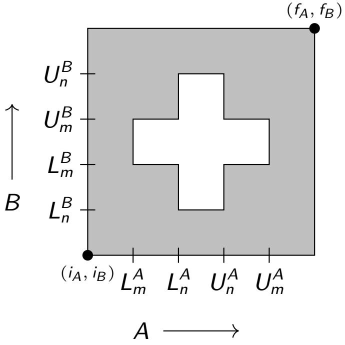

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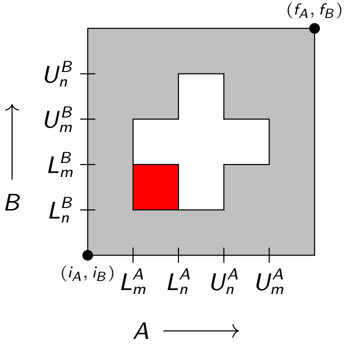

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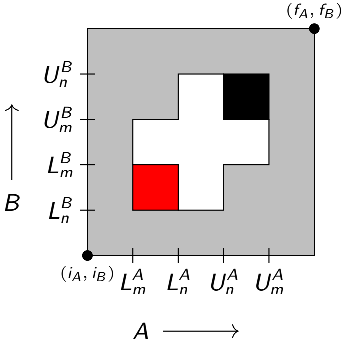

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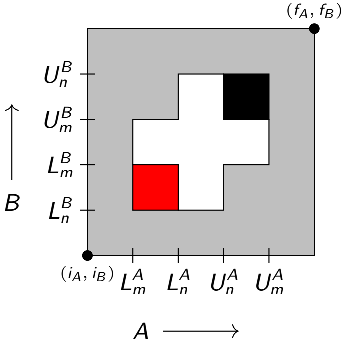

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- which can be locked (L) or unlocked (U) by each process

Fundamental questions:

- Which states are safe? (Predicate $S(x)$ on X^{op}.)
- Which states are reachable? (Predicate $R(x)$ on X.)

Outline

Rules for hom: core and op

$$\frac{T \ \textsc{type}}{T^{\mathsf{core}} \ \textsc{type}}$$

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$$\frac{T \ \textsc{type} \qquad t : T^{\mathsf{core}}}{it : T}$$

$$\frac{T \ \textsc{type} \qquad t : T^{\mathsf{core}}}{i^{\mathsf{op}} t : T^{\mathsf{op}}}$$

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$$\frac{T \ \ \textsc{type} \qquad s : T^{\mathsf{op}} \qquad t : T}{\hom_T(s,t) \ \ \textsc{type}}$$

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Id formation

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$$\frac{T \quad \text{TYPE} \qquad t : T}{r_t : \mathrm{Id}_T(t, t) \quad \text{TYPE}}$$

Rules for hom: right elimination and computation

$$\frac{T \ \ \text{TYPE} \qquad s : T^{\mathrm{core}}, t : T, f : \hom_T(i^{\mathrm{op}} s, t) \vdash D(f) \ \ \text{TYPE} \qquad s : T^{\mathrm{core}} \vdash d(s) : D(1_s)}{\begin{array}{c} s : T^{\mathrm{core}}, t : T, f : \hom_T(i^{\mathrm{op}} s, t) \vdash e_R(d, f) : D(f) \\ s : T^{\mathrm{core}} \vdash e_R(d, 1_s) \equiv d(s) : D(1_s) \end{array}}$$

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Rules for hom: left elimination and computation

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Syntactic results

- Transport: for a dependent type $t : T \vdash S(t)$:

$$t : T^{\mathrm{core}}, t' : T, f : \mathrm{hom}_T(i^{\mathrm{op}} t, t'), s : S(it) \\ \vdash \mathrm{transport}_{\mathrm{R}}(s, f) : S(t')$$

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- Composition: for a type T:

$$r : T^{\mathrm{op}}, s : T^{\mathrm{core}}, t : T, f : \hom_T(r, is), g : \hom_T(i^{\mathrm{op}} s, t) \vdash \mathrm{comp}_{\mathrm{R}}(f, g) : \hom_T(r, t)$$

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- With Σ types, we can define

$$\mathit{Reachable}(T) := \Sigma_{x : T} \hom_T(i, x)$$

$$\mathit{Safe}(T) := \Sigma_{x : T^{\mathrm{op}}} \hom_T(x, f)$$

for any type T with terms $i : T^{\mathrm{op}}, f : T$.

Outline

The interpretation

- Use the framework of comprehension categories
- Dependent types are represented by functors $T : \Gamma \to Cat$.
- Dependent terms are represented by natural transformations

$$\begin{tikzcd} \Gamma \arrow[r, bend left, "*"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow t"] & Cat \end{tikzcd}$$

  where $* : \Gamma \to Cat$ is the functor which takes everything to the one-object category.
- Context extension is represented by the Grothendieck construction which takes each functor $T : \Gamma \to Cat$ to the Grothendieck opfibration

$$\pi_\Gamma : \int_\Gamma T \to \Gamma.$$

Interpreting core and op in the empty context

$$\frac{T \ \text{TYPE}}{T^{\text{core}} \ \text{TYPE} \qquad T^{\text{op}} \ \text{TYPE}} \qquad\qquad \frac{T \ \text{TYPE} \qquad t : T^{\text{core}}}{it : T \qquad i^{\text{op}} t : T^{\text{op}}}$$

For any category T,

- $T^{\text{core}} := \text{ob}(T)$
- $T^{\text{op}} := T^{\text{op}}$
- $i : T^{\text{core}} \to T$ and $i^{\text{op}} : T^{\text{core}} \to T^{\text{op}}$ are the identity on objects.

Interpreting hom formation and introduction

$$\frac{T \ \ \text{TYPE} \qquad s : T^{\text{op}} \qquad t : T}{\text{hom}_T(s, t) \ \ \text{TYPE}} \qquad\qquad \frac{T \ \ \text{TYPE} \qquad t : T^{\text{core}}}{1_t : \text{hom}_T(i^{\text{op}} t, it) \ \ \text{TYPE}}$$

For any category T,

- Take the functor

$$\text{hom} : T^{\text{op}} \times T \to Set \hookrightarrow Cat.$$

- Take the natural transformation

$$\begin{tikzcd} T^{\text{core}} \arrow[r, bend left, "*"] \arrow[r, bend right, "\text{hom} \circ (i^{\text{op}} \times i)"'] \arrow[r, phantom, "\Downarrow 1_\bullet"] & Cat \end{tikzcd}$$

where each component $1_t : * \to \text{hom}(t, t)$ picks out the identity morphism of t.

Interpreting right hom elimination and computation

$$\frac{T \ \ \textsc{type} \qquad s : T^{\mathrm{core}}, t : T, f : \mathrm{hom}_T(i^{\mathrm{op}}s, t) \vdash D(f) \ \ \textsc{type} \qquad s : T^{\mathrm{core}} \vdash d(s) : D(1_s)}{\begin{array}{c} s : T^{\mathrm{core}}, t : T, f : \mathrm{hom}_T(i^{\mathrm{op}}s, t) \vdash e_R(d, f) : D(f) \\ s : T^{\mathrm{core}} \vdash e_R(d, 1_s) \equiv d(s) : D(1_s) \end{array}}$$

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\begin{tikzcd}
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T^{\text{core}} \arrow[uu, hook, "1_\bullet"] \arrow[ur, bend left, "*"] \arrow[ur, bend right, "D"'] \arrow[ur, phantom, "\Downarrow d"] &
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& & Cat \\
& & \\
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- Set $e_R(d)_{(s,t,f)} := D(1_s, f) d_{(s,s,1_s)}$

Interpreting left hom elimination and computation

$$\frac{T \quad \text{TYPE} \qquad s : T^{\mathrm{op}}, t : T^{\mathrm{core}}, f : \hom_T(s, it) \vdash D(f) \quad \text{TYPE} \qquad s : T^{\mathrm{core}} \vdash d(s) : D(1_s)}{s : T^{\mathrm{op}}, t : T^{\mathrm{core}}, f : \hom_T(s, it) \vdash e_L(d, f) : D(f) \qquad s : T^{\mathrm{core}} \vdash e_L(d, 1_s) \equiv d(s) : D(1_s)}$$

- Replace T by T^{op} and apply right hom elimination and computation.

Outline

A homotopical perspective

While the homotopy theory of isomorphisms in categories

$$\mathcal{C} \to \mathcal{C}^{(\cong)} \to \mathcal{C} \times \mathcal{C}$$

provides an interpretation of Martin-Löf's identity type, the homotopy theory of morphisms in categories

$$\mathcal{C} \to \mathcal{C}^{(\rightarrow)} \to \mathcal{C} \times \mathcal{C}$$

provides an interpretation of this hom former.

The weak factorization system

- Let $(\cong)$ denote the category with two objects and one isomorphism between them.
- Let $(\rightarrow)$ denote the category with two objects and one morphism between them.
- Then factorize the codiagonal of the one-point category in two ways

$$* + * \;\to\; (\cong) \;\to\; * \qquad\qquad * + * \;\to\; (\rightarrow) \;\to\; *$$

- which produces a factorization of any diagonal in two ways which each generate weak factorization systems.

$$\mathcal{C} \to \mathcal{C}^{(\cong)} \to \mathcal{C} \times \mathcal{C} \qquad\qquad \mathcal{C} \to \mathcal{C}^{(\rightarrow)} \to \mathcal{C} \times \mathcal{C}$$

- The first gives an interpretation of the Id type in *Cat*.
- The second underlies this interpretation of the hom type in *Cat*.

The weak factorization system continued

- The right class of this weak factorization system are those functors $p : E \to B$ which have the enriched right lifting property

$$\begin{tikzcd} * \arrow[r] \arrow[d, "\mathrm{DOM}"'] & E \arrow[d, "p"] \\ (\to) \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

- so all Grothendieck opfibrations (dependent projections) are in the right class.
- The functor $1_\bullet : T^{\text{core}} \hookrightarrow \int_{T^{\text{core}} \times T} \hom$ is the left part of the factorization of

$$i : T^{\text{core}} \to T.$$

- Then the right hom elimination and computation rule arises from the weak factorization system.

$$\begin{tikzcd} T^{\text{core}} \arrow[r, "d"] \arrow[d, hook, "1_\bullet"'] & \int_{\int_{T^{\text{core}} \times T} \hom} D \arrow[d, "\pi"] \\ \int_{T^{\text{core}} \times T} \hom \arrow[r, equal] \arrow[ur, dashed, "e_R(d)"] & \int_{T^{\text{core}} \times T} \hom \end{tikzcd}$$

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Summary & future work

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- integrate this into traditional Martin-Löf type theory
  - integrate Id and hom in the same theory
  - specify Σ, Π, etc

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We have:

- a directed type theory
- with a model in *Cat*.

Future work

We need to:

- integrate this into traditional Martin-Löf type theory
  - integrate Id and hom in the same theory
  - specify Σ, Π, etc
- find interpretations in categories of directed spaces
  - build 'directed' weak factorization systems
  - build universes

Thank you!

Further Reading

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