Skew monoidal structures on categories of algebras

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Skew monoidal categories

A version of monoidal categories (Szlachányi (2012))

Structural transformations need not be invertible:

$$\alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$
$$\lambda : I \otimes A \to A$$
$$\rho : A \to A \otimes I$$

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Example

- For $\mathcal{C}$ with coproducts, $(X/\mathcal{C})$ with

$$(X \xrightarrow{a} A) \oplus (X \xrightarrow{b} B) := X \xrightarrow{\mathrm{inl}} X + X \xrightarrow{a+b} A + B$$

- For $\mathcal{C}$ cocomplete, $[\mathcal{J}, \mathcal{C}]$ with unit J and tensor $F \star G := (\mathrm{lan}_J F) \circ G$ $\big($Altenkirch *et al.* (2010)$\big)$.

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Recently studied very actively (*list not exhaustive!*):

- Coherence properties: Lack & Street (2014), Andrianopoulos (2017), Bourke (2017), Uustalu (2017, 2018), ...
- Extensions, theory and examples: Street (2013), Campbell (2018), ...

Past work

Linton ('69), Kock ('71a, '71b), Guitart ('80), Jacobs ('94), Seal ('13), ...

$\mathcal{C}$ monoidal $\mathbb{T}$ a monoidal monad reflexive coequalizers in $\mathcal{C}$ + preservation conditions	$\Rightarrow$	$\mathcal{C}^{\mathbb{T}}$ monoidal

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$\mathcal{C}$ monoidal

$\mathbb{T}$ a monoidal monad $\Rightarrow$ $\mathcal{C}^{\mathbb{T}}$ monoidal

reflexive coequalizers in $\mathcal{C}$ +
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This work

$\mathcal{C}$ skew monoidal

$\mathbb{T}$ a strong monad $\Rightarrow$ $\mathcal{C}^{\mathbb{T}}$ skew monoidal

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monoids are T-monoids

Monoidal case ($\mathcal{C}, \mathbb{T}$ monoidal)

Definition (Kock (1971))

For $(A, a), (B, b), (C, c) \in \mathcal{C}^{\mathbb{T}}$ a map $h : A \otimes B \to C$ in $\mathcal{C}$ is *bilinear* if it is linear in each argument:

$$
\begin{array}{ccccccc}
T(A) \otimes B & \xrightarrow{T(A)\otimes\eta} & T(A) \otimes T(B) & \xrightarrow{\kappa} & T(A \otimes B) & \xrightarrow{Th} & TC \\
{\scriptstyle a\otimes B}\downarrow & & & & & & \downarrow{\scriptstyle c} \\
A \otimes B & & & \xrightarrow[h]{\qquad\qquad\qquad} & & & C
\end{array}
$$

$$
\begin{array}{ccccccc}
A \otimes T(B) & \xrightarrow{\eta\otimes T(B)} & T(A) \otimes T(B) & \xrightarrow{\kappa} & T(A \otimes B) & \xrightarrow{Th} & TC \\
{\scriptstyle A\otimes b}\downarrow & & & & & & \downarrow{\scriptstyle c} \\
A \otimes B & & & \xrightarrow[h]{\qquad\qquad\qquad} & & & C
\end{array}
$$

Monoidal case ($\mathcal{C}$, $\mathbb{T}$ monoidal)

Aim

Construct $(-) \star (=) : \mathcal{C}^{\mathbb{T}} \times \mathcal{C}^{\mathbb{T}} \to \mathcal{C}^{\mathbb{T}}$ satisfying

1. $\mathcal{C}^{\mathbb{T}}(A \star B, C) \cong \mathrm{Bilin}_{\mathcal{C}}(A, B; C)$
2. A suitable preservation property to guarantee coherence

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2. A suitable preservation property to guarantee coherence

Construction (Linton 1969)

Reflexive coequalizer in $\mathcal{C}^{\mathbb{T}}$:

$$T\big(T(A) \otimes T(B)\big) \underset{T(a \otimes b)}{\overset{T\kappa}{\rightrightarrows}} T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B) \overset{\text{coeq.}}{\dashrightarrow} A \star B$$

NB: $U : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates reflexive coequalizers if T preserves them

Monoidal case ($\mathcal{C}, \mathbb{T}$ monoidal)

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Construct $(-) \star (=) : \mathcal{C}^{\mathbb{T}} \times \mathcal{C}^{\mathbb{T}} \to \mathcal{C}^{\mathbb{T}}$ satisfying

1. $\mathcal{C}^{\mathbb{T}}(A \star B, C) \cong \mathrm{Bilin}_{\mathcal{C}}(A, B; C)$
2. if every $(-) \otimes X$ and $X \otimes (-)$ preserve reflexive coequalizers, so do $(-) \star (A, a)$ and $(A, a) \star (-)$

Construction (Linton 1969)

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NB: $U : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates reflexive coequalizers if T preserves them

Monoidal case ($\mathcal{C}$, $\mathbb{T}$ monoidal)

Proposition (Guitart ('80), Seal ('13))

Suppose that

- $\mathcal{C}$ *has all reflexive coequalizers,*
- T *preserves reflexive coequalizers,*
- *Every* $(-) \otimes X$ *and* $X \otimes (-)$ *preserves reflexive coequalizers*

Then $(\mathcal{C}^{\mathbb{T}}, \star, TI)$ *is a monoidal category.*

Other versions are available: *e.g.* closed, symmetric, cartesian...

Skew monoidal case ($\mathcal{C}$ skew monoidal, $\mathbb{T}$ strong)

Classify left-linear maps

Construct an action $\mathcal{C}^{\mathbb{T}} \times \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$

Extend to a skew monoidal structure on $\mathcal{C}^{\mathbb{T}}$

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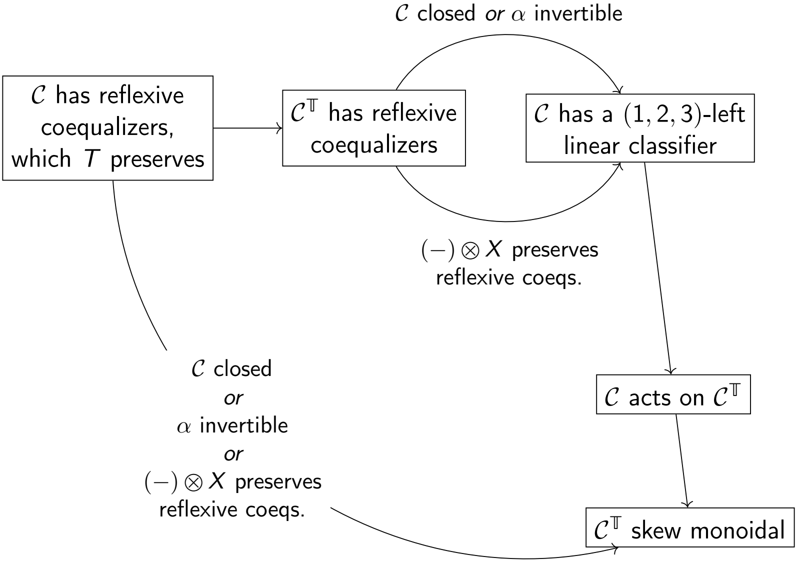
Extend to a skew monoidal structure on $\mathcal{C}^{\mathbb{T}}$

Background assumption:
$\mathcal{C}$ skew monoidal, $\mathbb{T}$ strong $\big(\mathrm{st} : T(A) \otimes B \to T(A \otimes B)\big)$

Factoring the proof



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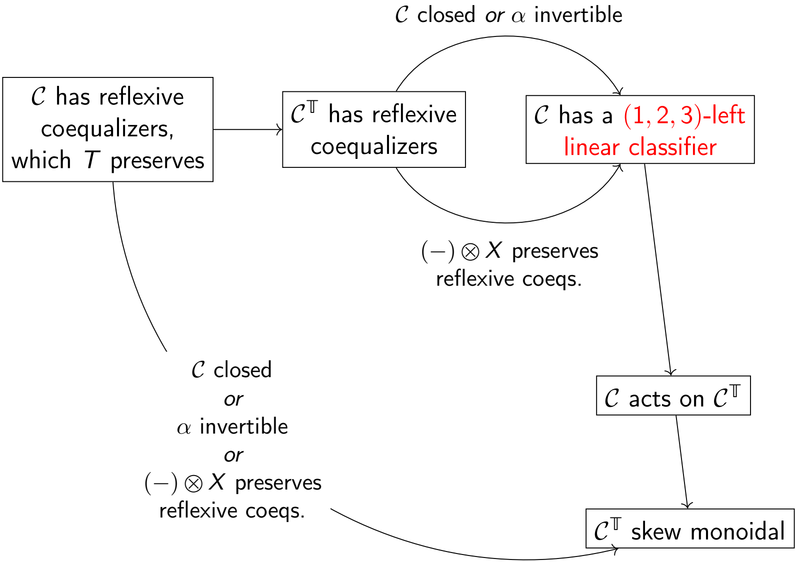

Left-linear maps

Definition (*c.f.* Kock (1971))

For $(A, a), (B, b) \in \mathcal{C}^{\mathbb{T}}$ and $P \in \mathcal{C}$, a map $h : A \otimes P \to C$ is *left linear* if

$$\begin{array}{ccccc} T(A) \otimes P & \xrightarrow{\mathrm{st}_{A,B}} & T(A \otimes P) & \xrightarrow{Th} & TB \\ {\scriptstyle a \otimes P}\downarrow & & & & \downarrow{\scriptstyle b} \\ A \otimes P & & \xrightarrow[h]{\qquad\qquad} & & B \end{array}$$

Definition (*c.f.* Guitart ('80), Jacobs ('94), Seal ('13))

A *left-linear classifier* is a family of maps $\sigma_{A,P} : A \otimes P \to A \star P$ such that

1. $(A \star P, \tau_{A,P}) \in \mathcal{C}^{\mathbb{T}}$
2. $\sigma_{A,B}$ is left-linear,
3. Every left-linear map $A \otimes P \to B$ factors uniquely:

$$\begin{array}{ccc} A \otimes P & \xrightarrow{\sigma} & A \star P \\ & \searrow^{\forall \text{ left-linear maps}} & \Big\downarrow{\scriptstyle \exists! \text{ algebra map}} \\ & & B \end{array}$$

Determines an isomorphism $\mathcal{C}^{\mathbb{T}}(A \star P, B) \cong \mathrm{LeftLin}_{\mathcal{C}}(A, P; B)$.

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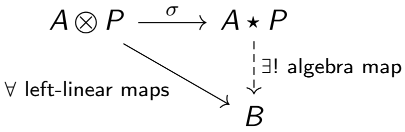

Determines an isomorphism $\mathcal{C}^{\mathbb{T}}(A \star P, B) \cong \mathrm{LeftLin}_{\mathcal{C}}(A, P; B)$.

$\rightsquigarrow$ Need to build in a *preservation property* to guarantee coherence

Definition

For $(A, a), (B, b) \in \mathcal{C}^{\mathbb{T}}$ and $P_1, \ldots, P_n \in \mathcal{C}$, a map

$$h : \left(\cdots \left((A \otimes P_1) \otimes P_2 \right) \cdots \otimes P_{n-1} \right) \otimes P_n \to B$$

is *n-left linear* if

$$\begin{array}{ccccc} T(A) \otimes P_1 \otimes \cdots \otimes P_n & \xrightarrow{\mathrm{st}^{\otimes n}} & T(A \otimes P_1 \otimes \cdots \otimes P_n) & \xrightarrow{Th} & TB \\ {\scriptstyle a \otimes P_1 \otimes \cdots \otimes P_n} \downarrow & & & & \downarrow {\scriptstyle b} \\ A \otimes P_1 \otimes \cdots \otimes P_n & & \xrightarrow[h]{\hspace{4em}} & & B \end{array}$$

where $\mathrm{st}^{\otimes 1} := \mathrm{st}$ and $\mathrm{st}^{\otimes (n+1)} := \mathrm{st} \circ \mathrm{st}^{\otimes n}$.

n-left linear maps

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$\rightsquigarrow$ An n-parameter version of left-linearity.

n-left linear classifiers

Definition

A *n-left linear classifier* is a family of maps $\sigma_{A,P_1} : A \otimes P_1 \to A \star P_1$ such that

1. $(A \star P_1, \tau_{A,P_1}) \in \mathcal{C}^{\mathbb{T}}$
2. $\sigma_{A,B}$ is left-linear,
3. Every n-left linear map $\big(\cdots\big((A \otimes P_1) \otimes P_2\big)\cdots\big) \otimes P_n \to B$ factors uniquely:

$$\begin{array}{ccc}
A \otimes P_1 \otimes \cdots \otimes P_n & \xrightarrow{\sigma \otimes P_2 \otimes \cdots \otimes P_n} & (A \star P_1) \otimes P_2 \otimes \cdots \otimes P_n \\
& \searrow{\scriptstyle \forall\ n\text{-left linear}} & \Big\downarrow{\scriptstyle \exists!\ (n-1)\text{-left linear map}} \\
& & B
\end{array}$$

A $(1, \ldots, n)$*-left linear classifier* is a 1-left linear classifier that is also an i-left linear classifier $(1 \leqslant i \leqslant n)$.

n-left linear classifiers

Lemma

If $h : \big(\cdots \big((A \otimes P_1) \otimes P_2 \big) \cdots \big) \otimes P_{n+1} \to B$ *is* $(n+1)$*-left linear, then (if they exist)*

1. *The transpose* $\tilde{h} : A \otimes P_1 \otimes \cdots \otimes P_n \to [P_{n+1}, B]$ *is n-left linear,*
2. $h \circ \alpha^{-1} : (A \otimes P_1 \cdots \otimes P_{n-1}) \otimes (P_n \otimes P_{n+1}) \to B$ *is n-left linear*

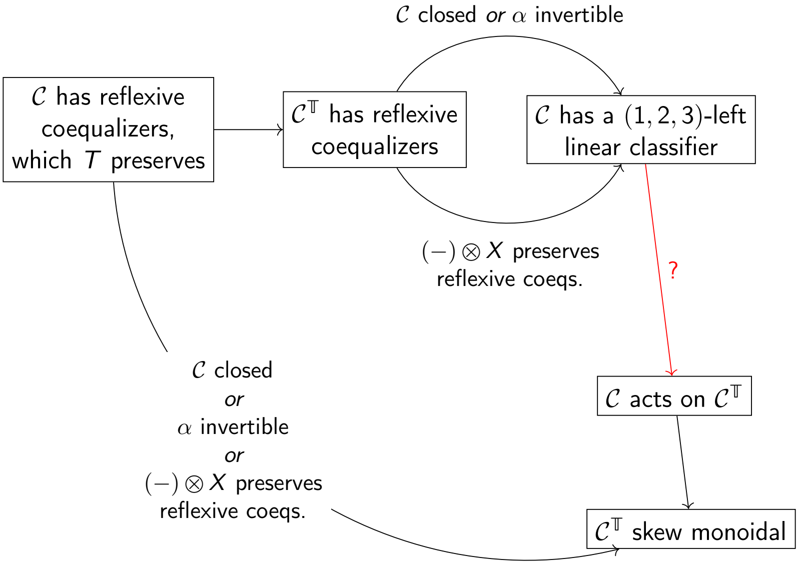
Lemma

If $\mathcal{C}$ *has an n-left linear classifier and satisfies either*

- $\mathcal{C}$ *is closed, or*
- α *is invertible*

Then $\mathcal{C}$ *has an* $(n+1)$*-left linear classifier.*

Factoring the proof



From classifier to action

Proposition

If $\mathcal{C}$ has a $(1,2,3)$-left linear classifier $\sigma_{A,B} : A \otimes B \to A \star B$, then

1. $\star : \mathcal{C}^{\mathbb{T}} \times \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$ *is a skew action, and*
2. *The free-forgetful adjunction $F : \mathcal{C} \leftrightarrows \mathcal{C}^{\mathbb{T}} : U$ is strong.*

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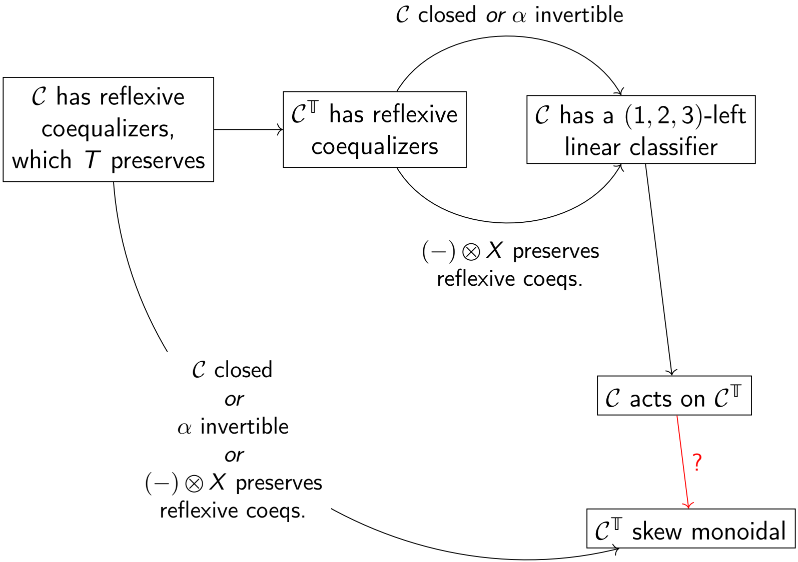
Holds in particular if $\mathcal{C}$ has a 1-left linear classifier and

- $\mathcal{C}$ is closed, *or*
- α is invertible

Factoring the proof



Proposition

Given

1. *A skew monoidal category* $(\mathcal{C}, \otimes, I)$,
2. *A category* $\mathcal{A}$,
3. *A skew action* $\star : \mathcal{A} \times \mathcal{C} \to \mathcal{A}$,
4. *A strong adjunction* $(U, \mathrm{st}^U) : \mathcal{A} \leftrightarrows \mathcal{C} : (F, \mathrm{st}^F)$

Then, setting

$$A \circledast B := A \star UB$$

makes $(\mathcal{A}, \bar{\star}, FI)$ *a skew monoidal category.*

From classifier to skew monoidal

Proposition

If $\mathcal{C}$ has any of

1. *A $(1,2,3)$-left linear classifier $A \otimes B \to A \star B$,*
2. *A 1-left linear classifier $A \otimes B \to A \star B$, and $\mathcal{C}$ is closed,*
3. *A 1-left linear classifier $A \otimes B \to A \star B$, and α is invertible*

Then $(\mathcal{C}^{\mathbb{T}}, \star, TI)$ is skew monoidal.

From classifier to skew monoidal

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If $\mathcal{C}$ has any of

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Then $(\mathcal{C}^{\mathbb{T}}, \star, TI)$ is skew monoidal.

Question: how do we construct a $(1,2,3)$-left linear classifier?

Constructing a left-linear classifier

Construction

Reflexive coequalizer in $\mathcal{C}^{\mathbb{T}}$:

$$T\big(T(A) \otimes P\big) \underset{T(a \otimes P)}{\overset{T\mathrm{st}}{\rightrightarrows}} T^2(A \otimes P) \xrightarrow{\mu} T(A \otimes P) \overset{\text{coeq.}}{\dashrightarrow} A \star P$$

Then

1. $\mathcal{C}^{\mathbb{T}}(A \star P, B) \cong \mathrm{LeftLin}_{\mathcal{C}}(A, P; B)$,
2. If $T(- \otimes X)$ preserves reflexive coequalizers,
   get a $(1, 2, 3)$-left linear classifier.

Constructing a left-linear classifier

Proposition

If $\mathcal{C}$ has all reflexive coequalizers, T preserves reflexive coequalizers, and any of the following:

1. *Every $(-) \otimes P$ preserves reflexive coequalizers,*
2. *$\mathcal{C}$ is closed,*
3. *α is invertible*

Then $\mathcal{C}$ has a $(1,2,3)$-left linear classifier:

$$A \otimes B \xrightarrow{\eta} T(A \otimes B) \xrightarrow{\text{coeq.}} A \star B$$

Putting it all together

Theorem

If $\mathcal{C}$ has all reflexive coequalizers, T preserves reflexive coequalizers, and any of the following:

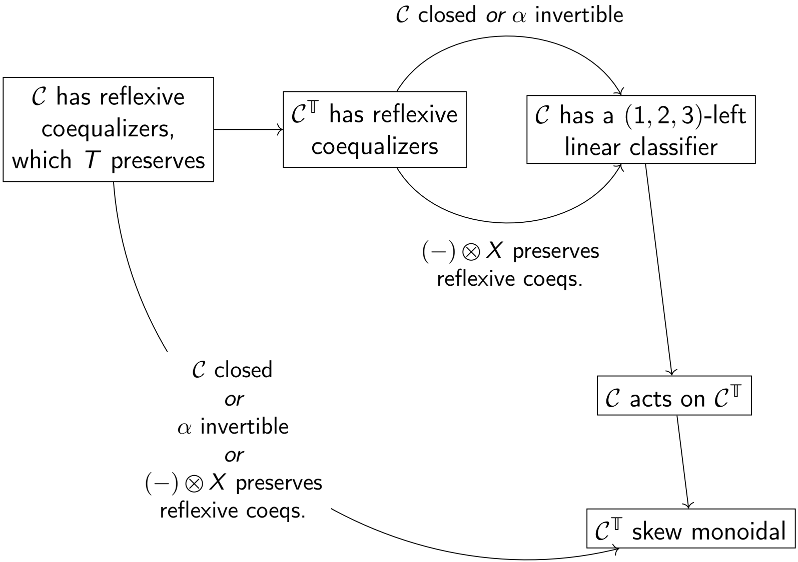
1. *Every $(-) \otimes P$ preserves reflexive coequalizers,*
2. *$\mathcal{C}$ is closed,*
3. *α is invertible*

Then $(\mathcal{C}^{\mathbb{T}}, \star, TI)$ is skew monoidal.

Remark

Can also do the calculation directly — but it is much more intricate! (*c.f.* Seal (2013))

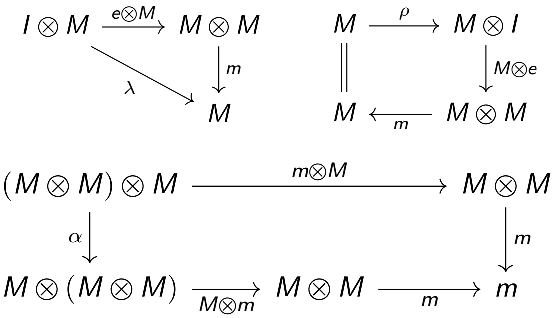
Factoring the proof



Monoids in skew monoidal categories

Definition

A *monoid* in $\mathcal{C}$ is an object M with $(I \xrightarrow{e} M \xleftarrow{m} M \otimes M)$ such that



Question: how do we construct free monoids?

Free monoids as initial algebras

Lemma (folklore)

Let $(\mathcal{C}, \otimes, I)$ *be a monoidal category with finite coproducts* $(0, +)$ *and* ω*-colimits, and* $X \in \mathcal{C}$ *such that*

1. *Every* $(-) \otimes P$ *preserves coproducts and* ω*-colimits, and*
2. $X \otimes (-)$ *preserves* ω*-colimits*

Then the initial $(I + X \otimes -)$*-algebra is the free monoid on* X*.*

Free monoids as initial algebras

Lemma

Let $(\mathcal{C}, \otimes, I)$ be a skew monoidal category with finite coproducts $(0, +)$ and ω-colimits, and $X \in \mathcal{C}$ such that

1. *Every $(-) \otimes P$ preserves coproducts and ω-colimits, and*
2. *$X \otimes (-)$ preserves ω-colimits*

Then the initial $(I + X \otimes -)$-algebra is the free monoid on X.

Free monoids as colimits: $(\mathcal{C}, \otimes, I)$ monoidal

Lemma (Dubuc (1974), Melliès (2008), Lack (2008))

There exists a monoidal category $\mathcal{P}$ such that

$$\mathbf{MonCat}_{\mathrm{strong}}(\mathcal{P}, \mathcal{C}) \simeq (I/\mathcal{C})$$

Lemma (Dubuc (1974), Melliès (2008), Lack (2008))

For $(I \xrightarrow{x} X) \in (I/\mathcal{C})$, if

1. *$\mathcal{C}$ has $\mathcal{P}$-colimits, and*
2. *Every $(-) \otimes C$ and $C \otimes (-)$ preserves $\mathcal{P}$-colimits*

Then $\operatorname{colim} \mathrm{D}_x$ is the free monoid on $(I \xrightarrow{x} X)$, for $\mathrm{D}_x : \mathcal{P} \to \mathcal{C}$ the monoidal functor corresponding to $(I \xrightarrow{x} X)$.

Free monoids as colimits: $(\mathcal{C}, \otimes, I)$ skew monoidal

Lemma

There exists a skew monoidal $\mathcal{P}$ such that

$$\mathbf{SkMonCat}_{\text{strong}}(\mathcal{P}, \mathcal{C}) \simeq (I/\mathcal{C})$$

Lemma

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Monoids in $(\mathcal{C}^{\mathbb{T}}, \star, TI)$ as T-monoids

Definition (*c.f.* Fiore *et al.* (1999))

For a strong monad $(\mathbb{T}, \mathrm{st})$, a *$T$-monoid* is an object $M \in \mathcal{C}$ with

1. A monoid structure $(M \otimes M \xrightarrow{m} M \xleftarrow{e} I)$,
2. An algebra structure (M, τ_M),

Such that the multiplication $m : M \otimes M \to M$ is a left-linear map.

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Example

If $\mathcal{C}$ has two monoidal structures $(\otimes, I)$ and $(\bullet, J)$ related by a *distributivity structure*, then for $\mathbb{T}$ the free $\bullet$-monoid monad on $\mathcal{C}$, a T-monoid in $(\mathcal{C}, \otimes, I)$ is a *near semiring object* (Fiore 2016, Fiore & S. 2017).

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Such that the multiplication $m : M \otimes M \to M$ is a left-linear map.

Proposition

If $\mathcal{C}$ has a $(1,2,3)$-left linear classifier $\sigma_{A,B} : A \otimes B \to A \star B$, then

$$T\text{-}Mon\big((\mathcal{C}, \otimes, I)\big) \cong Mon\big((\mathcal{C}^{\mathbb{T}}, \star, TI)\big)$$

Monoids in $(\mathcal{C}^{\mathbb{T}}, \star, TI)$ as T-monoids

Monoidal examples

1. If $\mathcal{C}$ has finite coproducts,

$$\mathcal{C}^{\mathbb{T}} \cong T\text{-}Mon\big((\mathcal{C}, +, 0)\big) \cong Mon(\mathcal{C}^{\mathbb{T}})$$

2. For $M \in Mon(\mathcal{C})$ and $M^{\otimes} := \big(M \otimes (-), m \otimes (-), e \otimes (-)\big)$

$$\big(M/Mon(\mathcal{C})\big) \cong M^{\otimes}\text{-}Mon(\mathcal{C}) \cong Mon(\mathcal{C}^{\mathbb{M}^{\otimes}})$$

(Fiore & S. 2017).

Summary & contribution

Adapted classical construction of monoidal structure on $\mathcal{C}^{\mathbb{T}}$ to skew monoidal setting.

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Monoids in $(\mathcal{C}^{\mathbb{T}}, \star, TI)$ are T-monoids in $(\mathcal{C}, \otimes, I)$.

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$\rightsquigarrow$ Associated paper in preparation.